Solvability Conditions for the Vector Lexicographic Optimization Problems

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***Abstract*—Vector problems with the lexicographic principle of optimality are formulated and investigated. We have revealed conditions of existence of solutions of multi-criteria problems of lexicographic optimization with an unbounded feasible set on the basis of applying properties of a recession cone of a convex feasible set, the cone which puts in order a feasible set lexicographically with respect to optimization criteria.**

Keywords*—*lexicographic optimization, vector criterion, existence of solutions, Pareto-optimal solutions

I. Introduction

Among vector problems, lexicographic problems form a fairly wide and important class of optimization problems [1-3]. The lexicographic approach to solving multicriteria problems consists in a strict ranking of criteria in terms of relative importance and allows you to optimize a more important criterion at the expense of any losses for all other, less important criteria. Most often, such multicriteria problems arise when additional criteria are successively introduced into ordinary scalar optimization problems, which may not have a unique solution.

The aim of the research presented in this article is to establish conditions for the solvability of multicriteria lexicographic optimization problems with an unbounded admissible set based on the use of the properties of a recessive cone of a convex feasible set [4], a cone lexicographically ordering the feasible set with respect to optimization criteria [5-6]. In this paper, we present new results that continue the research reflected, in particular, in [6–9] and concerning the existence of solutions to vector problems of lexicographic optimization.

II. Problem Statement

In the criterion space, we introduce a binary relation of the lexicographic order between vectors $z = (z_1, z_2, \dots, z_\ell)$ and $z' = (z'_1, z'_2, \dots, z'_\ell)$ such that $z \geq^L z' \Leftrightarrow (z = z') \vee \left(\exists j \in N_\ell : \forall i \in N_{j-1} \left(z_j > z'_j,\ z_i = z'_i\right)\right)$, where $N_0 = \emptyset$.

Let us consider a lexicographic optimization problem of the following type: $Z_L(F, X)$: $max^L\{F(x) | x \in X\}$, where $F(x) = \left(f_1(x), \dots, f_\ell(x)\right)$, $\ell \geq 2, f_k(x) = \langle c_k, x\rangle$, $c_k \in R^n$, $k \in N_\ell = \{1,2,\dots,\ell\}$, $X = \left\{x \in R^n \middle| g^i(x) \leq 0, x \geq 0,\ i \in N_m\right\}$, $X \neq \emptyset$, $g^i(x)$, $i \in N_m$ are convex functions. In the problem of lexicographic optimization, particular criteria are ordered by importance. This gives rise to the concept of the lexicographic optimum.

By solving the problem $Z_L(F, X)$ we mean the search for elements of the set $L(F, X)$ of lexicographic optimal solutions, which we define in this way:

$$L(F, X) = \{x \in X | v(x, F, X) = \emptyset\},$$

where $v(x, F, X) = \left\{x' \in X | \exists j \in N_\ell : f_j(x') > f_j(x) \wedge\ j = \right.$ $\left. = min\{i \in N_\ell :\ f_i(x') \neq f_i(x)\}\right\}$.

It follows directly from the definition of lexicographically optimal solutions that the set $L(F, X)$ can also be specified using recurrence relations. Thus,

$$L_i(F, X) = Argmax\{\, f_i(x) : x \in L_{i-1}(F, X),\ i \in N_\ell \quad (1)$$

where $Arg\, max\{\cdot\}$ is a set of all optimal solutions to the corresponding maximization problem, $L_0(F, X) = X$, $L_\ell(F, X) = L(F, X)$.

It follows from relations (1) that the inclusions of the sequence of sets

$$X \supseteq L_1(F, X) \supseteq L_2(F, X) \supseteq \ldots \supseteq L_\ell(F, X) = L(F, X),$$

is a true, it means each next particular criterion narrows the set of solutions obtained taking into account all the previous particular criteria. As it is known [1, 2], a set $L(F, X)$ can be defined as the result of solving a sequence of ℓ scalar convex programming problems $Z_{L_i}(F, X)$, $i \in N_\ell$. So, the problem $Z_L(F, X)$ can be viewed as a sequential optimization problem. Let us note the important properties of problems $Z_{L_i}(F, X)$, $i \in N_\ell$, [7]: any local minimum (maximum) is a global minimum (maximum).

Definition 1. A solution $x^* \in X$ to a problem $Z_L(F, X)$ will be called lexicographically optimal if it is not worse than any other admissible solution $y \in X$ in understanding the relation $\geq^L$, that is, if $F(x^*) - F(y) \geq^L 0$.

So, for an arbitrary $x \in X$, the assertion is true

$$x \in L(F, X) \Leftrightarrow \{y \in X | F(y) >^L F(x)\} = \emptyset.$$

In the lexicographic problem, optimizations achieve an arbitrarily small increase in a more important criterion at the expense of any losses in other less important criteria.

The validity of such properties follows from the definition of the lexicographically optimal solution of the problem $Z_L(F, X)$.

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Property 1. If for an feasible solution $x^0 \in X$ and $\forall\, x \in X \setminus \{x^0\}$ of the problem $Z_L(F,X)$ the inequality $f_1(x) < f_1(x^0)$ holds, then $x^0 \in L(F,X)$.

Property 2. If for an feasible solution $x \in X$ of the problem $Z_L(F,X)$ $\exists x' \in X\backslash\{x\}$ such that $f_1(x') > f_1(x)$, then $x \notin L(F,X)$.

III. Existence of Lexicographically Optimal Solutions

The solvability of the problem of finding lexicographically optimal solutions on a feasible set X and the structure of the set of optimal solutions depend on the properties of the order of the preference relation, the structure of the feasible domainX, the nature of its elements, properties of the vector function $F(x)$, etc.

According to [2], the finiteness of the set X is a sufficient condition for the existence of optimal solutions to the lexicographic problem optimization. Also, the set $L(F,X)$ is not empty if the set of vector estimates $Y = \{F(x) | x \in X\}$ is bounded and closed. However, in the case of an infinite feasible set X, the set of lexicographically optimal solutions may be empty.

It is topical to study the issues of solvability of lexicographic vector optimization problems in which the set of feasible solutions is not bounded and convex.

The unboundedness of a convex set X means that

$$0^{+}\mathrm{X}\backslash\{0\} \neq \emptyset,$$

where

$$0^{+}\mathrm{X} = \{y \in R^n | \forall x \in X : x + ty \in X, t \geq 0\}$$

is the recessive cone of the set X.

We will analyze the problem $Z_L(F,X)$taking into account the properties of the recessive cone $0^{+}\mathrm{X}$ [4] and the cone $K^L = \{x \in R^n | Cx >^L 0\}$ lexicographically ordering the admissible set with respect to the optimization criteria, which we will also call the cone of promising [5] lexicographic directions of the problem $Z_L(F,X)$, since the transition from any point $x_1 \in R^n$ to the point $x_2 = x_1 + y$ where $y$belongs tothe cone K^L leads to the inequality $Cx_2 >^L Cx_1$, that is, to the lexicographic increase in the values of the vector criterion of the problem.

The cone K^L that determines the lexicographic order in space R^ℓ is a convex cone of directions of lexicographically positive vectors and can be represented as a union of disjoint sets:

$$K^L = K_1 \cup K_2 \cup \ldots \cup K_\ell,$$

where

$$K_1 = \{x \in R^n | c_1 x > 0\},$$

$$K_2 = \{x \in R^n |\ c_1 x = 0,\ c_2 x > 0\},$$

$$\ldots,$$

$$K_\ell = \{x \in R^n | c_1 x = 0,\ c_2 x = 0, \ldots, c_{\ell-1} x = 0,\ c_\ell x > 0\}.$$

For an arbitrary $x \in X$, the statement [2] is true:

$$x \in L(F,X) \Leftrightarrow (x + K^L) \cap X = \emptyset. \quad (2)$$

For the problems of lexicographic optimization, we consider the necessary and sufficient conditions for the existence of lexicographically optimal solutions, which were started in work [2] and continued in [6-9].

In the case of a convex closed unbounded feasible set X of the problem $Z_L(F,X)$, the theorem is valid.

Theorem 1. A necessary condition for the existence of lexicographically optimal solutions to the problem $Z_L(F,X)$ is the empty intersection of the cone K^L of promising lexicographic directions and the recessive cone $0^{+}\mathrm{X}$, that is,

$$K^L \cap 0^{+}X = \emptyset. \quad (3)$$

Proof. Let us suppose by way of contradiction, that the set $L(F,X) \neq \emptyset$, but condition (3) is not satisfied, that is, the intersection of the cones K^L and $0^{+}X$ is not empty: $K^L \cap 0^{+}X \neq \emptyset$. Then the following relations are true:

$$(x + K^L) \cap X \supseteq (x + K^L) \cap (x + 0^{+}X) =$$

$= x + (K^L \cap 0^{+}X) \neq \emptyset$. Taking into account formula (2), we can conclude that the set $L(F,X) = \emptyset$. But this contradicts the condition of the theorem and thereby proves its validity.

The converse statement of the theorem is generally not true. In the monograph [2] an example is given in which condition (3) is satisfied for an feasible set X, but the set of its extreme points is unbounded, and as a result, the set $L(F,X) = \emptyset$.

The direction of the lexicographically positive vector will be called the lexicographically positive direction.

The theorem is true [2].

Theorem 2. Let V be a non-empty set of extreme points of a convex closed set. X. If V is a bounded set, then the set Xhas a lexicographic maximum if and only if it is bounded in all lexicographically positive directions.

In our notation, under the conditions of Theorem 2, the set $L(F,X)$ is not empty if and only if condition (3) is satisfied.

In the case of a convex, unbounded and polyhedral set, the corollary to Theorem 2 [2] is true.

Consequence. A closed convex polyhedral set X has a lexicographic maximum if and only if it is bounded in all lexicographically positive directions.

Theorem 1 and the corollary to Theorem 2 imply the following theorem.

Theorem 3. Let the feasible set X of the problem $Z_L(F,X)$ be a closed convex polyhedral set. A necessary and sufficient condition for the existence of lexicographically optimal solutions to this problem is the fulfillment of equality (3).

Note that the multifaceted condition of a convex closed unbounded set X is essential for the statement of the fact that condition (3) is a necessary and sufficient condition for the existence of lexicographically optimal solutions to the problem $Z_L(F,X)$.

IV. Existence of Lexicographically Optimal Solutions in Integer Optimization Problems

Let us now consider the integer problem of lexicographic optimization of the following form:

$$Z_L^I(F,X): \quad max^L\{F(x)|x \in X \cap Z^n\},$$

$X = \{x \in R^n | Ax \le b\}, X \ne \emptyset, Z^n \subset R^n, Z^n$ is the set of all integer vectors with R^n.

We denote by $X_I = \{x \in R^n | Ax \le b\}_I$ the convex hull of the integer vectors of the polyhedron set X. We will call it an integer hull X. It is obvious that $X_I \subseteq X$. If the set X is bounded, then the set X_I is also bounded.

Theorem 4 [10] is valid.

Theorem 4. For any rational polyhedral set X, its integer envelope X_I forms a rational polyhedral set.

Like linear lexicographic optimization problems, lexicographic linear integer optimization problems can have an empty feasible set or have an unbounded feasible set. For a given polyhedron, it seems difficult to find out whether its integer hull X_I is empty. However, if the feasible domain of the integer lexicographic problem is not empty, then the existence of its solutions can be checked by considering the linear relaxation of the integer lexicographic problem.

Statement 1[11]. Let $X = \{x \in R^n | Ax \le b\}$ be a rational polyhedron whose integer hull is not empty, and let $c \in R^n$ be some vector, not necessarily rational. Then the optimum $max\{cx | x \in X\}$ is bounded if and only if $max\{cx | x \in X_I\}$ is bounded.

Theorem 5. Let in the problem $Z_L^I(F,X)$, where $X = \{x \in R^n | Ax \le b\}$ be a rational polyhedral unbounded set whose integer hull is nonempty, and let K^L be a cone of lexicographic directions (not necessarily rational). Then the problem $Z_L(F,X_I)$ has lexicographic optimal solutions if and only if the problem $Z_L(F,X)$ has solutions.

V. Conclusion

The existence of solutions to convex and integer lexicographic optimization problems with linear criteria functions and an unbounded feasible set have been investigated. On the basis of the analysis of the specified problems, taking into account the properties of perspective lexicographic directed and recessive directed cones, the necessary and sufficient conditions for the existence of solutions to the investigated problems have been established. The obtained conditions can be successfully used in the development of algorithms for finding optimal solutions to these lexicographic optimization problems.

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