Martingale Representation and the Problem of Nonlinear Filtration

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***Abstract*—We study Brownian functionals, the conditional mathematical expectation of which with respect to natural filtration (the so-called filter) is not stochastically smooth from the point of view of their representability as a stochastic Itô integral with an explicit form of the integrand. The considered class of functionals also includes those that are not smooth in the sense of Malliavin, to which both the well-known Clark-Ocone formula (1984) and its generalization, the Glonti-Purtukhia representation (2017), are inapplicable.**

Keywords—Malliavin (stochastic) derivative, martingale representation, filtering problem, Clark-Ocone formula, Glonti-Purtukhia representation

I. Introduction

The problem of nonlinear filtering is as follows: we are interested in the estimation of a signal process ξ_t which cannot be observed directly but we have an observation process η_t which is related to ξ_t. The best estimate in the mean square sense of $f(\xi_t)$ based on the natural σ-algebra of observations $\Im_t^\eta = \sigma\{\eta_s: 0 \le s \le t\}$ is given by the conditional mathematical expectation $E[f(\xi_t)|\Im_t^\eta]$. In the General case this estimate depends nonlinearly on observations and is called a nonlinear filter. A practical and mathematically more appealing method for solving the filtering problem is to derive a stochastic differential equation for the filter and use Ito's stochastic calculus.

If ξ_t is the solution of a stochastic differential equation and f is a C^2-function, then according to the Ito's formula $f(\xi_t)$ is a semimartingale and hence, under appropriate conditions, $E[f(\xi_t)|\Im_t^\eta]$ is a right-continuous semimartingale with respect to σ-algebras $\Im_t^\eta$. Therefore, if every right-continuous L_2-martingale can be represented as a stochastic integral with respect to a Wiener process, then we can derive a stochastic differential equation for $E[f(\xi_t)|\Im_t^\eta]$. As is known, the central results of nonlinear filtering theory -- the derivation of the stochastic equations satisfied by the optimal nonlinear filter. Thus, the question of the stochastic integral representation of martingales is very important for filtering problems.

The stochastic integral representation theorem, also known as the martingale representation theorem, states that any square integrable Brownian functional is represented as a stochastic integral with respect to a Brownian Motion. The first proof of the martingale representation theorem was implicitly provided by Ito ([1]) himself. Indeed, here it is proved that any square integrable Wiener functional can be expressed as a series of multiple stochastic integrals, further it is shown that a multiple integral can be expressed as an iterated stochastic integral, and, as a result, a stochastic integral representation can be obtained from here.

Many years later, Dellacherie ([3]) gave a simple new proof of Ito's theorem using Hilbert space techniques. Many other articles were written afterward on this problem and its applications but one of the pioneer work on explicit descriptions of the integrand is certainly the one by Clark ([2]).

On the other hand, in the 80s of the last century (Harrison and Pliska [6]), it became clear that martingale representation theorems (along with Girsanov's absolutely continuous change of measure theorem) play an important role in modern financial mathematics. Therefore, the following question naturally arises: can any $\Im_t^\eta$-martingale be represented as a stochastic integral? It turned out that we have a positive answer to this question (Clark ([2])) when $\Im_t^\eta = \Im_t^w$ (where w_t is a Wiener process), but in general this is not so. This is shown in the example of Kallianpur ([5]) (to whom M. Ior described it, and the latter, in turn, attributes the example to H. Kunita): let (w_t^1, w_t^2) be a Wiener process in R^2, and let

$$M_t = \int_0^t w_s^1 \mathrm{d}w_s^2.$$

It's obvious that

$$N_t = (w_t^1)^2 - 1 = 2\int_0^t w_s^1 \mathrm{d}w_s^1$$

is a $\Im_t^M$ - martingale, but it cannot be represented as a stochastic integral with respect to M_t.

After Clark ([2]) obtained the formula for the stochastic integral representation for Brownian Motion functionals, many authors tried to explicitly find the integrand. The works of Haussmann ([4]), Ocone ([7]), Ocone and Karatzas (1991), Karatzas, Ocone and Li (1991), Shyriaev and Yor (2003), Graversen, Shyriaev and Yor (2006) and Renaud and Remillard (2007) are especially important in this direction.

Hence, taking into account the needs of modern financial mathematics, it is not enough to know only the existence of an integral representation, it is necessary to be able to find the explicit form of the integrand of the integral representation. It is known that for stochastically smooth functionals, the integrand is calculated by Ocone's formula ([7]), which was

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later generalized by Glonti and Purtukhia ([12]), when only the filter of the functional is stochastically smooth. Here we study functionals whose filter is no longer smooth and propose a method for finding the integrand.

We study the question of representing of Brownian functionals as a stochastic Itô integral with an explicit form of the integrand. The considered class of functionals also includes functionals that are not smooth in the sense of Malliavin, to which both the well-known Clark-Ocone formula ([7]) and its generalization, the Glonti-Purtukhia formula ([12]), are inapplicable.

II. AUXILIARY CONCEPTS AND RESULTS

Let a Brownian Motion $B = B_t, t \in [0, T]$, be given on a probability space $(\Omega, \Im, P)$, and let $\Im_t^B = \sigma\{B_s: 0 \le s \le t\}$. Let $C_p^\infty(R^n)$ be the set of all infinitely differentiable functions $f: R^n \to R$ such that f and all its partial derivatives have polynomial growth. Denote by Sm the class of smooth random variables F of the form

$$F = f(B_{t_1}, B_{t_2}, \dots, B_{t_n}), f \in C_p^\infty(R^n), t_i \in [0, T].$$

It is known that Sm is dense in $L_2(\Omega)$.

Definition 1. The stochastic derivative (derivative in the Malliavin sense) of a smooth random variable F is defined as a random process $D_t F$ defined by the relation (Nualart and Pardoux ([8]))

$$D_t F = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(B_{t_1}, B_{t_2}, \dots, B_{t_n}) I_{[0,t_i]}(t).$$

D is closable as an operator from $L_2(\Omega)$ to $L_2(\Omega, L_2([0, T]))$. Denote its domain of definition by $D_{1,2}$.

This means that $D_{1,2}$ is equal to the closure of the class of smooth random variables in the norm

$$||F||_{1,2} = \{E[F^2] + E[||DF||^2_{L_2([0,T])}]\}^{1/2}.$$

Theorem 1. (Clark-Ocone's representation formula, Ocone ([7]) If F is differentiable in the sense of Malliavin, $F \in D_{1,2}$, then the following integral representation holds

$$F = E[F] + \int_0^T E[D_t F | \Im_t^B] dB_t, \quad (P\text{–α.σ.}). \qquad (1)$$

Shiryaev and Yor (2003) and Graversen, Shiryaev and Yor (2006) proposed another method for finding the integrand based on the Itô formula and Levy's theorem for the Levy martingale $M_t = E[F | \Im_t^B]$ associated with the considered functional F (as F they considered the so-called "maximal" type functionals of Brownian Motion). Later, using the Clarke-Ocone formula, Renaud and Remillard (2007) established an explicit martingale representation for Brownian functionals, which also depend on the trajectory (in particular, here F is a continuously differentiable function of three smooth quantities: from the Brownian Motion with drift and processes of its maximum and minimum).

It is clear that the class of functionals to which the Clark-Ocone formula can be applied is limited by the condition that they must be Malliavin differentiable. We study questions of the stochastic integral representation of stochastically non-smooth functionals. Glonti, Jaoshvili and Purtukhia ([10], [11]) proposed a method for obtaining an integral representation for a non-smooth Brownian functionals of a special form using the Trotter-Meyer theorem, which establishes a connection between the predictable quadratic characteristic of a semimartingale and its local time.

Further, it turned out that the requirement for the smoothness of a functional can be weakened by the requirement for the smoothness of only its conditional mathematical expectation. It is known that if a random variable is stochastically differentiable in the sense of Malliavin, then its conditional mathematical expectation is also differentiable.

Proposition 1. (Nualart ([9]), Proposition 1.2.8) If $F \in D_{1,2}$, then $E(F | \Im_s^B) \in D_{1,2}$ and

$$D_t[E(F | \Im_s^B)] = E(D_t F | \Im_s^B) I_{[0,s]}(t).$$

On the other hand, the conditional mathematical expectation may be smooth even if the random variable is not stochastically smooth. For example, it is known that

$$E[I_{\{B_T \le c\}} | \Im_t^B] = \Phi\left(\frac{c - B_t}{\sqrt{T - t}}\right) \in D_{1,2},$$

where c is some real constant and Φ is the standard normal distribution function.

Remark 1. Here we have used the following statement from Nualart ([9], Proposition 1.2.6): The event indicator I_A is Malliavin differentiable if and only if the probability $P(A)$ equals zero or one.

Glonti and Purtukhia ([12]) generalized the Clark-Ocone formula to the case when the functional is not stochastically smooth, but its conditional mathematical expectation is stochastically differentiable, and proposed a method for finding the integrand.

Theorem 2. (Glonti-Purtukhia formula, ([12])) Assume that $G_t = E[F | \Im_t^B]$ is a Malliavin differentiable functional ($G_t(\cdot) \in D_{1,2}$) for almost all $t \in [0, T)$. Then the following stochastic integral representation is valid:

$$G_T = F = E[F] + \int_0^T \nu_u dB_u, \quad (P\text{–α.σ.}). \qquad (2)$$

where

$$\nu_u = \lim_{t \uparrow T} E[D_u G_t | \Im_u^B] \text{ ιν } L_2([0, T] \times \Omega).$$

Remark 2. It should be noted that the result of the above theorem can also be useful for smooth functionals (see Proposition 2 below).

Proposition 2. The smooth Brownian functional $F = B_T^+ := \max\{0, B_T\}$ have the following stochastic integral representation

$$B_T^+ = \sqrt{\frac{T}{2\pi}} + \int_0^T \Phi\left(\frac{B_s}{\sqrt{T-s}}\right) dB_s, \quad (P\text{-a.s.}).$$

Proof. It is easy to see that

$$E[B_T^+] = \sqrt{\frac{T}{2\pi}}.$$

Further, using the Glonti-Purtukhia representation (2), we have

$$G_t = E[B_T^+|\Im_t^B] = E\left[B_T I_{\{B_T>0\}}|\Im_t^B\right] =$$

$$= \frac{1}{\sqrt{2\pi(T-t)}}\int_0^\infty x\cdot\exp\left\{-\frac{(x-B_t)^2}{2(T-t)}\right\}dx.$$

Hence, due to the rule of stochastic differentiation and the standard integration technique, we obtain

$$D_s G_t = I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(T-t)}}\times$$

$$\times\int_0^\infty \frac{x(x-B_t)}{T-t}\cdot\exp\left\{-\frac{(x-B_t)^2}{2(T-t)}\right\}dx =$$

$$= I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(T-t)}}\times$$

$$\times\int_{-B_t/\sqrt{T-t}}^\infty x(x\sqrt{T-t}+B_t)\exp\left\{-\frac{x^2}{2}\right\}dx =$$

$$= I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(T-t)}}\times$$

$$\times\left[\sqrt{T-t}\int_{-B_t/\sqrt{T-t}}^\infty x d\left(\exp\left\{-\frac{x^2}{2}\right\}\right)+\right.$$
$$\left.B_t\int_{-B_t/\sqrt{T-t}}^\infty d\left(\exp\left\{-\frac{x^2}{2}\right\}\right)\right]=$$

$$= I_{[0,t]}(s)\Phi\left(\frac{B_t}{\sqrt{T-t}}\right).$$

Therefore

$$E[D_s G_t|\Im_s^B] = I_{[0,t]}(s)\frac{1}{\sqrt{2\pi(t-s)}}\times$$

$$\times\int_{-\infty}^\infty \Phi\left(\frac{x}{\sqrt{T-t}}\right)\cdot\exp\left\{-\frac{(x-B_s)^2}{2(t-s)}\right\}dx.$$

Now, using the relation

$$\lim_{t\to T}\Phi\left(\frac{x}{\sqrt{T-t}}\right) = \begin{cases}0, & x<0;\\ 1, & x>0\end{cases}$$

it is easy to check that

$$\lim_{t\to T} E[D_s G_t|\Im_s^B] = I_{[0,T]}(s)\frac{1}{\sqrt{2\pi(T-s)}}\times$$

$$\times\int_{-\infty}^\infty \lim_{t\to T}\left[\Phi\left(\frac{x+B_s}{\sqrt{T-t}}\right)\cdot\exp\left\{-\frac{x^2}{2(T-s)}\right\}\right]dx =$$

$$= I_{[0,T]}(s)\frac{1}{\sqrt{2\pi(T-s)}}\int_{-B_s}^\infty \exp\left\{-\frac{x^2}{2(T-s)}\right\}dx =$$

$$= I_{[0,T]}(s)\left[1-\Phi\left(-\frac{B_s}{\sqrt{T-s}}\right)\right] = I_{[0,T]}(s)\Phi\left(\frac{B_s}{\sqrt{T-s}}\right),$$

which, on the basis of Theorem 2, together with the above relations, completes the proof of the proposition.

III. MAIN RESULTS

Here we consider one class of Brownian functionals, which includes non-smooth functionals (therefore, it is impossible to use the well-known Clark-Ocone formula (1)), depending on the trajectory, and we propose a method for obtaining a constructive stochastic integral representation. In addition, the class under consideration also includes functionals for which even the conditional mathematical expectation is not stochastically smooth and, therefore, neither the generalization of the Clark-Ocone formula (2) is applicable to them.

In particular, we study the functional of integral type

$$\int_0^T h_s(\omega)ds,$$

where $h_s(\omega)\notin D_{1,2}$ but $E[h_s(\omega)|\Im_t^B]\in D_{1,2}$.

If $h_s(\omega)$ is not differentiable in the Malliavin sense, then the Lebesgue averaging (with respect to ds) is in general not differentiable in the Malliavin sense.

On the other hand, in this case, even the conditional mathematical expectation is not smooth, since we have:

$$E\left[\int_0^T h_s(\omega)ds\,|\Im_t^B\right] = \int_0^t h_s(\omega)ds + E\left[\int_t^T h_s(\omega)ds\,|\Im_t^B\right],$$

where the first term is not differentiable, but the second term is differentiable in the sense of Malliavin (it is known that if $G_s(\cdot)\in D_{1,2}$ for almost all s and $G_s(\omega)$ is Lebesgue integrable for a.e. ω, then

$$\int_t^T G_s(\omega)ds\in D_{1,2}).$$

Theorem 3. If $h(x)$ is a bounded measurable function on R, then the function $V(t,x) = E[\int_t^T h(B_s)ds\,|B_t = x]$ satisfies the requirements of the Ito formula and the following stochastic integral representation is valid

$$\int_0^T h(B_t)dt = \int_0^T E[h(B_t)\}dt + \int_0^T \frac{\partial}{\partial x}V(t,B_t)dB_t,\ (P\text{-a.s.}),$$

Proof. It is well known that for all measurable bounded functions g and $t<s$ we have

$$E[g(B_s)|\Im_t^B] = \int_{-\infty}^\infty g(y)p(t,s,B_t,dy),$$

where for any Borel subset A of $R=(-\infty,\infty)$: $p(t,s,B_t,A)=P[B_s\in A|\Im_t^B]$ the transition probability of Brownian motion and

$$p(t,s,x,A)=\frac{1}{\sqrt{2\pi(s-t)}}\int_A \exp\left\{-\frac{(x-y)^2}{2(s-t)}\right\}dy.$$

Therefore, using the well-known properties of conditional mathematical expectation and Brownian motion, we can write

$$V(t,x)=E\left[\int_t^T h(B_s)ds\,\middle|B_t\right]|_{B_t=x}=$$

$$=\left\{\int_t^T E[h(B_s)|B_t]ds\right\}|_{B_t=x}=\left\{\int_t^T E[h(B_s)|\Im_t^B]ds\right\}|_{B_t=x}=$$

$$=\left\{\int_t^T\left[\int_{-\infty}^{\infty}h(y)\frac{1}{\sqrt{2\pi(s-t)}}\exp\left\{-\frac{(B_t-y)^2}{2(s-t)}\right\}dy\ \right]ds\right\}|_{B_t=x}=$$

$$=\int_t^T\left\{\frac{1}{\sqrt{2\pi(s-t)}}\left[\int_{-\infty}^{\infty}h(y)\exp\left\{-\frac{(x-y)^2}{2(s-t)}\right\}\right]dy\right\}ds.$$

The last relation shows that, on the one hand, $V(t,x)$ is an integral with variable boundary with respect to t, and on the other hand, with respect to x, it is an integral that depends on a parameter. Therefore, it is easy to verify that in our case $V(t,x)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x, that is, $V(t,x)$satisfies the conditions of the Itô's formula.

According to Ito's formula, we have

$$V(t,B_t)=V(0,B_0)+\int_0^t\left[\frac{\partial}{\partial s}V(s,B_s)+\frac{1}{2}\frac{\partial^2}{\partial x^2}V(s,B_s)\right]ds+$$

$$+\int_0^t\frac{\partial}{\partial x}V(s,B_s)dB_s,\qquad (P\text{–α.σ.}).\qquad (3)$$

On the other hand, due to the Markov property of the Brownian Motion

$$V(t,B_t)=E\left[\int_t^T h(B_s)ds\,\middle|B_t=x\right]|_{x=B_t}=$$

$$=E\left[\int_t^T h(B_s)ds\,\middle|B_t\right]=E\left[\int_t^T h(B_s)ds\,\middle|\Im_t^B\right],\ \ (P\text{-a.s.}).$$

Therefore, under the conditions of the theorem, the process

$$\int_0^t h(B_s)ds+V(t,B_t)=\ E\left[\int_0^t h(B_s)ds\,\middle|\Im_t^B\right]+$$

$$+E\left[\int_t^T h(B_s)ds\,\middle|\Im_t^B\right]=E\left[\int_0^T h(B_s)ds\,\middle|\Im_t^B\right]:=M_t,$$

is a martingale.

Further, according to Levy's theorem, it is obvious that M_t is a continuous martingale. On the other hand, a continuous martingale of bounded variation starting from 0 is identically equal to 0. Therefore, in equality (3), the term of bounded variation in total with an additional term ($\int_0^t h(B_s)ds$) of bounded variation of martingale M is equal to zero.

Hence, taking into account equality

$$M_0=V(0,B_0)=E\left[\int_0^T h(B_s)ds\,\middle|B_0\right]=$$

$$=E\left[\int_0^T h(B_s)ds\,\middle|\Im_0^B\right]=E\left[\int_0^T h(B_s)ds\right],\quad (P\text{-a.s.}),$$

we easily complete the proof of the theorem.

Let $h_s(\omega)$ be an integrable process adapted to the flow of σ-algebras$\Im_s^B$. Denote

$$F(t,T)=\int_t^T h_s(\omega)ds$$

and

$$F=F(0,T)=\int_0^T h_s(\omega)ds.$$

Theorem 4. Suppose that $F(t,T)$ admits a decomposition $F(t,T)=F_1(t,T)+F_2(t,T)$, where $F_1(t,T)$ is a continuous process of finite variation, adapted to the flow of σ -algebras $\Im_t^B$ with $F_1(0,T)=0$ (if such a decomposition does not exist, then we assume that $F_1(t,T)\equiv 0$). If the function $V(t,x)=E[F_2(t,T)|B_t=x]$ satisfies the requirements of the classical Itô formula (i.e. $V(\cdot,\cdot)\in C^{1,2}([0,T\times R])$), then the following stochastic integral representation is fulfilled

$$F=E[F]+\int_0^T\frac{\partial}{\partial x}V(t,B_t)dB_t,\quad (P\text{–α.σ.}).\qquad (4)$$

Remark 3. It should be noted that the result of the above theorem is especially interesting for non-smooth $h_s(\omega)$, although it is also useful in the case of smooth $h_s(\omega)$.

Theorem 5. Let $h_s(\omega)\in D_{1,2}$ for almost all s. Then the Clark-Ocone representation (1) for the functional $F=\int_0^T h_s(\omega)ds$ follows from the above representation (4).

Remark 4. Note that an approach similar to our theorem can also be used for functionals depending on the last moment of time. There is the following statement.

Proposition 3. If $F=I_{\{B_T\le c\}}$, then the function $V(t,x)=E[F|B_t=x]$ satisfies the requirements of the Ito's formula and the following stochastic integral representation is valid

$$I_{\{B_T\le c\}}=\Phi\left(\frac{c}{\sqrt{T}}\right)+\int_0^T\frac{1}{\sqrt{T-t}}\varphi\left(\frac{c-B_t}{\sqrt{T-t}}\right)dB_t,\ \ (P\text{-a.s.}),$$

where φ is the density of the standard normal distribution.

Now consider the stochastically non-smooth integral functional

$$F=\int_0^T I_{\{B_t\le c\}}dt.\qquad (5)$$

Proposition 4. The functional F from (5) has the following stochastic integral representation

$$F=\int_0^T\Phi\left(\frac{c}{\sqrt{t}}\right)dt+\int_0^T\Psi(c,t,B_t\,)dB_t,$$

where

$$\Psi(c,t,x\,)=-2\sqrt{T-t}\varphi\left(\frac{c-x}{\sqrt{T-t}}\right)+$$

$$+(c-x)\left[2I_{\{x\le c\}}-1-erf\left(\frac{c-x}{\sqrt{2(T-t)}}\right)\right]$$

and

$$\mathrm{erf}\,(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp\{-u^2\}du.$$

Proposition 5. If $f(\cdot,\cdot):[0,T]\times R\to R$ is a measurable bounded function and $h_s(\omega)=f(s,B_s(\omega))$, then the function $V(t,x)=E[F(t,T)|B_t=x]$ satisfies the requirements of the Ito formula.

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REFERENCES

[1] K. Ito, “Multiple Wiener Integral,” Journal of the Mathematical Society of Japan, vol. 3(1), 1951, pp. 157-169.

[2] M.C. Clark, “The representation of functionals of Brownian motion by stochastic integrals,” J. Ann. Math. Stat., vol. 41, 1970, pp. 1282-1295.

[3] C. Dellacherie, “Intégrales stochastiques par rapport aux processus de Wiener ou de Poisson,” Séminaire de probabilités de Strasbourg, vol. 8, 1974, pp. 25-26.

[4] U. G. Haussmann, “On the integral representation of functionals of Ito processes,” Stochastics, vol. 3(1), 1979, pp. 17–27.

[5] G. Kallianpur, Stochastic filtering theory. Springer-Verlag, New York, 1980.

[6] J. M. Harrison and S. R. Pliska, “Martingales and stochastic integrals in the theory of continuous trading,” Stochastic Processes and Applications, vol. 11, 1981, pp. 215-260.

[7] D. Ocone, “Malliavin calculus and stochastic integral representation formulas of diffusion processes,” Stochastics, vol. 12, 1984, pp. 161-185.

[8] D. Nualart and E. Pardoux, “Stochastic calculus with anticipating integrands,” Probability Theory and Related Fields, vol. 78(4), 1988, pp. 535-581.

[9] D. Nualart, The Malliavin calculus and related topics. Springer-Verlag, Berlin, 2006.

[10] O. Glonti and O. Purtukhia, “Hedging of One European Option of Integral Type in Black-Scholes Model,” International Journal of Engineering and Innovative Technology (IJEIT), vol. 4(5), 2014, pp. 51-61.

[11] O. Glonti, V. Jaoshvili, and O. Purtukhia, “Hedging of European Option of Exotic Type,” Proceedings of A. Razmadze Mathematical Institute, vol. 168, 2015, pp. 25-40.

[12] O. Glonti, and O. Purtukhia, “On one integral representation of functionals of Brownian motion,” SIAM J. Theory of Probability and Its Applications, vol. 61 (1), 2017, pp. 133-139.