On the One Nonparametric Estimate of Poisson Regression Function

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***Abstract*—The limiting distribution of the integral square deviation of kernel-type nonparametric estimator of Poisson regression function is established. The test of the hypothesis testing about Poisson regression function is constructed. The question of consistency of the constructed test is studied. The power asymptotic of the constructed test is also studied for certain types of close alternatives.**

Keywords*—*Poisson regression function, power of the test, consistency, limiting distribution

Let random variable Y take values $0,1,2,\dots$ with probabilities

$$\Pi(k,\lambda) = \mathrm{P}\{Y=k\} = \frac{\lambda^k}{k!}e^{-\lambda}, \quad \lambda>0, \quad k=0,1,2,\dots .$$

Assume that the parameter λ is the function of an independent variable $x\in[0,1]$, i.e.

$$\mathbf{P}\{Y=k\} = \frac{\lambda^k(x)}{k!}e^{-\lambda(x)}.$$

$\lambda(x)$ is known as Poisson regression function (see [1], [2]). Let x_i, $i=1,2,\dots,n$, be the division points of the interval $[0,1]$:

$$x_i = \frac{2i-1}{2n}, \quad i=1,2,\dots,n.$$

Let further Y_i, $i=1,2,\dots,n$, be independent Poisson random variables with $\mathbf{P}\{Y_i=k|\ x_i\} = \Pi(k,\lambda(x_i))$. The problem consists in estimating the function $\lambda(x)$, $x\in[0,1]$, by sample $Y_1,Y_2,\dots,Y_n$ [1]. Problems of this kind arise, for example, in medicine [3], [4], in astrophysics [5] and so on.

As an estimator for $\lambda(x)$ we consider the following statistic (see [6], [7])

$$\hat{\lambda}_n(x) = \lambda_{1n}(x)\lambda_{2n}^{-1}(x),$$
$$\lambda_{\nu n}(x) = \frac{1}{nb_n}\sum_{i=1}^{n} K\left(\frac{x-x_i}{b_n}\right)Y_i^{2-\nu}, \quad \nu=1,2,$$

where $K(x)$ is some distribution density satisfying the requirements which we formulate in what follows, and $b_n\to 0$ is a sequence of positive numbers.

We assume that kernel $K(x)\ge 0$ is chosen so that it is a function with finite variation and satisfies the conditions $K(x)=K(-x)$, $K(x)=0$ as $|x|\ge\tau>0$, $\int K(x)dx=1$. The class of such functions is denoted by $H(\tau)$.

Let $C^{(i)}$ denotes the class of functions $\lambda(x)$, $x\in[0,1]$ having bounded derivatives up to order i, $i=1,2$.

We also introduce the notation

$$\bar{T}_n = nb_n\int_{\Omega_n(\tau)}\left[\lambda_{1n}(x)-\mathbf{E}\lambda_{1n}(x)\right]^2dx, \ \Omega_n(\tau)=[\tau b_n, 1-\tau b_n],$$
$$T_n = nb_n\int_{\Omega_n(\tau)}\left[\hat{\lambda}_n(x)-\lambda(x)\right]^2\lambda_{2n}^2(x)dx,$$
$$Q_{ij}=\psi_n(x_i,x_j), \quad \psi_n(u,v)=\int_{\Omega_n(\tau)} K\left(\frac{x-u}{b_n}\right)K\left(\frac{x-v}{b_n}\right)dx,$$
$$\sigma_n^2 = 4(nb_n)^{-2}\sum_{k=2}^{n}\lambda_k\sum_{i=1}^{k-1}\lambda_i Q_{ik}^2, \quad \lambda_i=\lambda(x_i), \ i=1,2,\dots.n,$$
$$\eta_{ij}^{(n)} = \frac{2\varepsilon_i\varepsilon_j Q_{ij}}{nb_n\sigma_n}, \quad \varepsilon_i = Y_i-\lambda(x_i),$$
$$\xi_k^{(n)} = \sum_{i=1}^{k-1}\eta_{ik}^{(n)}, \ k=2,\dots,n, \ \xi_1^{(n)}=0, \ \xi_k^{(n)}=0, \ k>n,$$
$$\mathrm{F}_k^{(n)} = \sigma(\omega:\ \varepsilon_1,\dots,\varepsilon_k),$$

where $\mathrm{F}_k^{(n)}$ is the σ-algebra generated by random variable $\varepsilon_1,\dots,\varepsilon_k$, $\mathrm{F}_0^{(n)}=(\Theta,\Omega)$ (in what follows, for the sake of simplicity, instead of $\xi_k^{(n)}$ and $\eta_{ij}^{(n)}$ we will write ξ_k and η_{ij}).

DOI: https://doi.org/10.54381/pci2023.03

Lemma 1. The stochastic sequence $(\xi_k, \mathrm{F}_k)_{k\geq 1}$ is a martingale-difference.

Lemma 2. Let $K(x)\in H(\tau)$ and $\lambda(x)$, $0\le x\le 1$, be also a function with bounded variation. If $nb_n\to\infty$, then

$$\frac{1}{nb_n}\sum_{i=1}^{n}K^{\nu_1}\left(\frac{x-x_i}{b_n}\right)K^{\nu_2}\left(\frac{y-x_i}{b_n}\right)\lambda^{\nu_3}(x_i)$$

$$=\frac{1}{b_n}\int_0^1 K^{\nu_1}\left(\frac{x-u}{b_n}\right)K^{\nu_2}\left(\frac{y-u}{b_n}\right)\lambda^{\nu_3}(u)\,du+O\left(\frac{1}{nb_n}\right),$$

uniformly in $x,y\in[0,1]$, $\nu_i\in \mathrm{N}\cup\{0\}$, $i=1,2,3$.

Lemma 3. Let $K(x)\in H(\tau)$ and $\lambda(x)\in C^{(1)}$ If $nb_n^2\to\infty$, then

$$b_n^{-1}\sigma_n^2\to\sigma^2(\lambda)=2\int_0^1\lambda^2(x)\,dx\int_{|x|\le 2\tau}K_0^2(x)\,dx$$

and

$$\Delta_n=\mathbf{E}\overline{T}_n=\Delta(\lambda)+O(b_n)+O\left(\frac{1}{nb_n}\right),$$

$$\Delta(\lambda)=\int_0^1\lambda(x)\,dx\int_{|x|\le\tau}K^2(x)\,dx,\quad K_0=K*K.$$

Theorem 1. Let $K(x)\in H(\tau)$ and $\lambda(x)\in C^{(1)}$ If $nb_n^2\to\infty$, then

$$\frac{b_n^{-1/2}\left(\overline{T}_n-\Delta(\lambda)\right)}{\sigma(\lambda)}\xrightarrow{d}N(0,1)$$

where $\Delta(\lambda)$ and $\sigma(\lambda)$ are defined as in Lemma 3 and $\xrightarrow{d}$ denotes convergence in distribution, and $N(0,1)$ is a random variable having standard normal distribution $\Phi(x)$.

Theorem 2. Let $K(x)\in H(\tau)$ and $\lambda(x)\in C^{(2)}$. Moreover, if $nb_n^2\to\infty$ and $nb_n^4\to 0$, then

$$b_n^{-1/2}\frac{T_n-\Delta(\lambda)}{\sigma(\lambda)}\xrightarrow{d}N(0,1)$$

The assertion of Theorem 2 allows us to construct the test of asymptotic level α, $0<\alpha<1$ for testing hypothesis H_0, according to which $\lambda(x)=\lambda_0(x)$, $x\in\Omega_n(\tau)$. The critical region is defined by the inequality

$$T_n\ge q_n(\alpha), \tag{1}$$

Where

$$q_n(\alpha)=\Delta(\lambda_0)+\lambda_\alpha\sqrt{b_n}\,\sigma(\lambda_0)$$

$$\Delta(\lambda_0)=\int_0^1\lambda_0(x)\,dx\int_{|u|\le\tau}K^2(u)\,du,$$

$$\sigma^2(\lambda_0)=2\int_0^1\lambda_0^2(x)\,dx\int_{|u|\le 2\tau}K_0^2(u)\,du,$$

and z_α is defined by the equality $\Phi(z_\alpha)=1-\alpha$.

Now let us investigate the asymptotic property of the test (1) (i.e. the behavior of the power function as $n\to\infty$). In first place, we consider the question of whether the test is consistent. The following assertion is true.

Theorem 3. Let all the conditions of Theorem 2 be fulfilled. Then as $n\to\infty$

$$\Pi_n(\lambda)=\mathbf{P}_{H_1}\{T_n\ge q_n(\alpha)\}\to 1$$

i.e. the test defined in (1) is consistent against any alternative

$$H_1:\ \lambda(x)\ne\lambda_0(x),\ 0\le x\le 1.$$

Thus for any fixed alternative the power of the test based on T_n tends to 1. However, if with a change of n the alternative changes converging to the basic Hypothesis H_0, then the power of the test will no longer necessarily converge to 1. Let us consider, for example, the sequence of Pitmen-type alternatives that are close to hypothesis H_0:

$$H_1:\ \lambda_1^{(n)}(x)=\lambda_0(x)+\gamma_n\varphi(x)+o(\gamma_n),\ \gamma_n\to 0$$

Theorem 4. Let $\lambda_0(x),\varphi(x)\in C^{(2)}$ and $K(x)\in H(\tau)$. If $b_n=n^{-\delta}$, $\gamma_n=n^{-1/2+\delta/4}$, $1/4<\delta<1/2$, then statistic $b_n^{-1/2}\left(\overline{T}_n-\Delta(\lambda_0)\right)\sigma^{-1}(\lambda_0)$ for alternative H_1 distributed in limit normally with parameters

$$\left(\frac{1}{\sigma(\lambda_0)}\int_0^1\varphi^2(x)\,dx,1\right)$$

i.e. the limiting power of the test is equal to

$$1-\Phi\left(\lambda_\alpha-\frac{1}{\sigma(\lambda_0)}\int_0^1\varphi^2(u)\,du\right).$$

Remark 1. It should be emphasized that the estimator $\hat{\lambda}_n(x)$ behaves worse near the boundary of the interval $[0,1]$

than in the interior interval $\left[\tau b_n, 1-\tau b_n\right]$ (see [8]). We therefore consider the integral square deviation on $\Omega_n(\tau)$ in order to avoid difficulties connected with this boundary effect; however it can be shown that in the conditions of Theorems 1 and 2 the results obtained above are also valid for the modified estimator (see [8], [9]) of the function $\lambda(x)$.

Remark 2. The idea of proof of Theorem 1 is analogous of proof Theorem 1 from paper [10].

Remark 3. Let x_i be the division points of the interval $[0,1]$ chosen so that relation $H(x_j)=\frac{2j-1}{2n}$, $j=1,\ldots,n$, where $H(x)=\int_0^x h(u)\,du$, $h(u)$ is some known continuous distribution density on $[0,1]$. Then, arguing analogously to the above, one can obtain a generalization of the results of this paper.

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