Existence of Solution of Fuzzy Differential Equation

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***Abstract*— In this paper, first a space of fuzzy numbers is constructed and a scalar product is introduced. The derivative of fuzzy function in this space is defined. Further, a fuzzy differential equation is considered and existence theorem for it's solution is proved.**

***Keywords*— Fuzzy numbers; linear space; derivative of the fuzzy function; differential equation; existence theorem**

I. Introduction

To investigate the fuzzy differential equation at first one has to introduce the definition of the derivative of the fuzzy function. This definition must allow one to investigate the fuzzy differential equation both theoretically and numerically.

For the first time, Chang and Zadeh introduced the concept of the fuzzy derivative in 1972, Dobous and Prade used the principle of continuity in its approach over ten years later. At the same time, Puri and Ralescu used the concept of H-Differentiability to extend the differential-valued functions. This led Seikkala to introduce the concept of the fuzzy derivative as an extension of the Hukuhara's derivative and fuzzy integral, which was the same as that proposed by Dobois and Prade.

In present work the linear space of pair fuzzy numbers is introduced. This linear space defines the metric between fuzzy numbers. Using this metric the derivative of the fuzzy function is determined. This allows us to give existence theorem of the fuzzy differential equation.

II. Space of the Fuzzy Numbers

Let's define by F the class of convex normal fuzzy numbers. Then for any $a \in F$ the set of α -cut of fuzzy number a is defined as the interval $a^\alpha = [L_a(\alpha), R_a(\alpha)]$, $\alpha \in [0,1]$, ([1]). Let $a \in F, b \in F$ and $a^\alpha = [L_a(\alpha), R_a(\alpha)]$, $b^\alpha = [L_b(\alpha), R_b(\alpha)]$. Then α -cut of fuzzy numbers $a+b$ and ka, $k \geq 0$, are defined as

$$a^\alpha + b^\alpha = [L_a(\alpha) + L_b(\alpha), R_a(\alpha) + R_b(\alpha)]$$

and

$$ka^\alpha = [kL_a(\alpha), kR_a(\alpha)],$$

respectively.

Note that F is not a linear space (the operation of subtraction is not defined in F).

We consider the set of pairs $(a,b) \in F \times F$ and d

efine the operation of addition, multiplication and equivalency as

$$\begin{aligned} &(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2+b_2),\\ &k\cdot(a,b)=(ka,\ \ kb), \quad k\geq 0\\ &(-1)\cdot(a,b)=(b,a)\\ &(a_1,a_2)\approx(b_1,b_2) <=> a_1+b_2=a_2+b_1 \end{aligned} \qquad (1)$$

The pair $(0,0)$ is taken as a zero element of this space i.e. the set of elements $(a,\ a),\ a \in F$.

The set of all pairs $(a,b) \in F \times F$ forms a structure of a linear space. Let

$$x = (a_1, a_2) \in F \times F, \quad y = (b_1, b_2) \in F \times F .$$

Then

$$a_i^\alpha = [L_{a_i}(\alpha), R_{a_i}(\alpha)],$$

$$b_i^\alpha = [L_{b_i}(\alpha), R_{b_i}(\alpha)],\ \alpha \in [0,1] .$$

For any $x, y \in F \times F$ the scalar product is defined as

$$x \circ y = \frac{1}{2}\int_0^1 [(L_{a_1}(\alpha) - L_{a_2}(\alpha))(L_{b_1}(\alpha) - L_{b_2}(\alpha)) +$$

$$+(R_{a_1}(\alpha) - R_{a_2}(\alpha))(R_{b_1}(\alpha) - R_{b_2}(\alpha)]d\alpha \qquad (2)$$

It may be shown that this definition satisfies all requirements of the scalar product. We denote this space by LF . Norm in this space is defined as

$$\| x \|^2 = \frac{1}{2}\int_0^1 [(L_{a_1}(\alpha) - L_{a_2}(\alpha))^2 + (R_{a_1}(\alpha) - R_{a_2}(\alpha))^2]d\alpha \qquad (3)$$

If x, y are vectors, i.e. $x = [x_1, x_2, ..., x_n],\ \ y = [y_1, y_2, ..., y_n]$, where $x_i, y_i \in F \times F$, then we define the scalar product and norm as follows

$$x \circ y = x_1 \circ y_1 + x_2 \circ y_2 + \ldots + x_n \circ y_n ,$$

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 + \ldots + \|x_2\|^2 . \qquad (4)$$

For the sake of simplicity we will write $x \in F \times F$ instead $x \in F^{(n)} \times F^{(n)}$.

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III. Derivative of the Fuzzy Function

Now, let's consider the fuzzy function $f(t) \in F$ for each $t \in [t_0, t_1]$ and define a derivative of the function $f(t)$.

For any $\alpha \in [0,1]$,

$$f_\alpha(t) = [L_{f(t)}(\alpha), R_{f(t)}(\alpha)], \alpha \in [0,1] \qquad (5)$$

is called α -cut of the function $f(t)$.

Definition. Let there exists such $\varphi(t) \in F$, $\psi(t) \in F$, $t \in [t_0, t_1]$, that

$$\lim_{\Delta t \to 0} \frac{(f(t+\Delta t),0) - (f(t),0)}{\Delta t} = (\varphi(t), \psi(t)) \qquad (6)$$

Then the pair $(\varphi(t), \psi(t)) \in F \times F$ is called a derivative of the function $f(t)$ at the point $t \in (t_0, t_1)$.

Now, let $f(t)$ be a pair of fuzzy functions, i.e.

$$f(t) = (f_1(t), f_2(t), \ \forall t \in (t_0, t_1).$$

From relation

$$f(t) = (f_1(t),0) + (0, f_2(t)) = f_1(t),0) - (f_2(t),0),$$

we see, that the derivative of the function $f(t)$ also is a pair from $F \times F$.

For any $\eta = \eta(t) \in F \times F$, which $\eta'(t) \in F \times F$, $t \in [t_0, t_1]$, let's consider the scalar product $f'(t) \circ \eta(t)$ defined by the formulae (2). It can be shown that

$$\int_t^T f'(\tau) \circ \eta(\tau) d\tau = f(\tau) \circ \eta(\tau) /_t^T - \int_t^T f(\tau) \circ \eta'(\tau) d\tau$$

$$\forall t, T \in [t_0, t_1]. \qquad (7)$$

IV. Statement of the Problem and Main Results

Let's consider the system differential equation

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t), \ t \in [0,T], \quad x(0) = x_0, \qquad (8)$$

Here $x(t)$ is n dimensional vector of phase coordinates of the object, $f(t)$ is m dimensional vector function, $A(t)$, $B(t)$ are $n \times n$ and $n \times m$ dimensional matrix functions. Assume that all these matrices are continuous on $[0,T]$ and $f(t), x_0$ are fuzzy, i.e. $x_0 \in F$, $f(t) \in F$, $\forall t \in [0,T]$. The solution of the problem (8) also will be fuzzy taking into account that $f(t), x_0$ are fuzzy.

It is necessary to note that investigation of the existence of the fuzzy solution for the problem (8) is very difficult.

Denote by $W = W(t)$ a fundamental matrix of the equation

$$\frac{dx(t)}{dt} = A(t)x(t). \qquad (9)$$

The function $W = W(t)$ is not fuzzy function. Let's take $H(t,\xi) = W(t) \cdot W^{-1}(\xi)$.

Theorem. Let function $f(t)$ and the vector x_0 be fuzzy and $H(t,\xi)$ is not negative. Then there exists a unique solution of problem (8) from F.

Proof. For any segment $[m, M]$, $m \le M$, let's define the support function

$$P(s) = \begin{cases} Ms, & s \ge 0 \\ ms, & s < 0. \end{cases}$$

This function is convex and positive homogeneous on $s \in R$, i.e. $P(\lambda s) = \lambda P(s)$, $\lambda \ge 0$. It is clear that $M = p(1)$, $m = -p(-1)$, i.e. using definition of the support function the problem may be written in the following form

$$\frac{\partial P_{x(t)}(\alpha, s)}{\partial t} = A(t) P_{x(t)}(\alpha, s) + P_{f(t)}(\alpha, s), \quad 0 < t \le T, \qquad (10)$$

$$P_{x(0)}(\alpha, s) = P_{x_0}(\alpha, s), \quad 0 \le \alpha \le 1, \ s \in R \qquad (11)$$

Consider the function

$$P(\alpha, s; t) = H(t,0) P_{x_0}(\alpha, s) + \int_0^t H(t,\xi) P_{f(\xi)}(\alpha, s) d\xi \qquad (12)$$

Checking we can see, that $P(\alpha, s; t)$ is a solution of the problem (10), (11). Really is (12) we get

$$P(\alpha, s; 0) = H(0,0) P_{x_0}(\alpha, s).$$

Also it is clear

$$\frac{\partial P(\alpha, s; t)}{\partial t} = \frac{\partial H(t,0)}{\partial t} P_{x_0}(\alpha, s) + H(t,t) P_{f(t)}(\alpha, s) + + \int_0^t \frac{\partial H(t,\xi)}{\partial t} P_{f(\xi)}(\alpha, s) d\xi$$

Considering that $P(\alpha, s; t)$ is solution of the problem

$$\frac{\partial P(\alpha, s; t)}{\partial t} = A(t) P(\alpha, s; t) + P_{f(t)}(\alpha, s), \ 0 < t \le T, \qquad (13)$$

$$P(\alpha, s; 0) = P_{x_0}(\alpha, s), \quad 0 \le \alpha \le 1, \ s \in R \qquad (14)$$

We see that $P(\alpha, s; t)$ is positive homogeneous on $s \in R$. From conditions $x_0 \in F$, $f \in F$, $t \in [0,T]$ we obtain that functions $P_{x_0}(\alpha, s)$ and $P_{f(t)}(\alpha, s)$ are convex.

Let's show that any components of the function $P(\alpha, s; t)$ are convex on $s \in R$.

It is clear for any $a(t) \ge 0$, the

$$\int_0^t a(\xi) P_{f(\xi)}(\alpha, s) d\xi$$

is convex on s. Really considering that $P_{f(\xi)}(\alpha, s)$ is convex on $s \in R$, then for $s_1, s_2 \in R$ we will have

$$\int_0^t a(\xi) P_{f(\xi)}\left(\frac{s_1 + s_2}{2}\right) d\xi \le \int_0^t a(\xi)\left[\frac{1}{2} P_{f(\xi)}(s_1) + \frac{1}{2} P_{f(\xi)}(s_2)\right] ds \le$$

$$\le \frac{1}{2}\int_0^t a(\xi) P_{f(\xi)}(s_1) d\xi + \frac{1}{2}\int_0^t a(\xi) P_{f(\xi)}(s_2) d\xi .$$

Taking into account that $H(x, \xi)$ is not negative from last relation we see that $P(\alpha, s; t)$ is positive homogeneous and convex. Then there is the segments $\lfloor L_{x(t)}(\alpha), R_{x(t)}(\alpha) \rfloor$, for which $x(t)$ is α -cut for this segments. Theorem is proved.

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