Interactive System for Solving Vector Optimization Problems and Its Applications

Marina Khoroshko
Cybernetics Institute of ANAS, Baku, Azerbaijan
khoroshko@yahoo.com

***Abstract*— In the paper a developed interactive system for vector optimization is described. The Interactive system is a software package that includes a variety of approaches and methods for solving vector optimization problems. User-friendly interface provides an efficient user operation within the system.**

Keywords*— vector optimization; multi criteria optimization; Pareto set; decision-making problems; decision makers (DM); uniformly distributed sequences (UDS)

I. Introduction

In practice, the solution of most optimization problems arising in the design of complex engineering systems and management, suggests the presence of more than one objective function. At the same time, identifying and accounting for dependencies between the individual processes and phenomena that previously were considered independently, leads to a sharp increase in the difficulty of obtaining reasonable guaranteed solutions. In such situations, software packages that implement a variety of approaches and methods, each of which is effective at a certain stage of solving the problem and for a certain class of problems are of practical interest. The most appropriate in this case is the organization of work in an interactive mode, allowing quickly solving the tasks. The developed interactive system allows the user to actively participate in the process of solving the problem. A typical scenario is to choose the approaches to problem solving, choose a scheme for solving the problems, switch from one scheme to another, adjust settings for the selected scheme, change the values of criteria limits, change the initial point, investigate the current point, and finally choose the type of presentation of search results.

II. Statement of the Problem

In general, the vector optimization problem is as follows: the system behavior is characterized by n-dimensional vector $x = \{x_1, x_2, ..., x_n\}, x \in X \subset E^n$ and is estimated by l-dimensional function $f(x) = \{f_1(x),\ f_2(x), ..., f_l(x)\}$, its components are given real functions of the variable x. Need to determine the point $x^* \in X$, which in some sense optimizes the value function $f_1(x), f_2(x), ..., f_l(x)$. Many methods for vector optimization problems in the decision-making problems does not always provide the best solution, because of containing subjective assessment. In this connection, it is most expedient to obtain the Pareto set of solutions of the problem, using the methods of vector optimization, and then make an objective choice of the preferred solution. In practice, the preferred solution is usually chosen by the decision makers (DM) [5, 6].

The purpose of developing an interactive system of vector optimization was to design software for obtaining a set of feasible solutions of optimization problems taking in account several criteria of optimality and choosing the best solution (or most acceptable) . To do this, different approaches to solving multi-criteria problems were included in the system:

1. an approach based on the idea of a convolution of vector criteria of quality
2. an approach based on the idea of finding a point of a given area;
3. an approach that implements UDS search;
4. an approach that implements a method for constructing Pareto sets using Peano-type curves;
5. an approach that provides a continuous method of constructing Pareto sets.

III. Implementation of the Proposed Approaches for Solving the Problems of Vector Optimization

A subsystem based on the idea of convolution of vector criterion of quality requires DM to give a vector of criteria preferences. It was implemented several options for setting vector of preferences, types of normalization of scalar criteria, a number of types of reduction. The system has a number of programms for solving constrained and unconstrained optimization problems.

The next subsystem implements an approach based on finding a point of a given area, limited with criteria constraints area [1]. In many real practical problems there is a case where a decision is to be made that satisfies certain constraints. This is considered sufficient, although the solution can be improved. This is the case, for example, in operational management tasks included in the software of the automated control systems. In such cases, the time for decision making is limited. The implemented approach is based on sequentially obtaining from decision-maker information cocerning to the desired values of criterion functions and the use of a method of finding the points of the feasible area [1]. The essence of this approach will now be described.

We consider the finite-dimensional problem of vector optimization:

$$F(x) = (f_1(x),...,f_l(x)) \to \min_{x \in D \subset R^n}, \quad (1)$$

$$D = \{x \in R^n : g_i(x) \le 0, i = 1,...,m\}, \quad (2)$$

where the given functions $f_j(x), g_j(x)$ are continuously differentiable, convex, $D \subset R^n$ is a non-empty bounded admissible set.

The approach, in which the solution of the problem (1), (2) is carried out iteratively in an interactive mode is as follows. At each s-th iteration the user sets the desired values of criterion functions f_i^s, $i = 1,...,l$. To find $x^s \in D$, satisfying conditions

$$f_i(x) \le f_i^s, i = 1,...,l, x \in D, \quad (3)$$

the sequence $\{x_q^s\}, q = 0,1,...,$ $x_0^s = x^{s-1}$ - the solution taken at the previous iteration, x_0^0 - an initial admissible point of the admissible set, is constructed:

$$F_q(x_q^s; x_{x-1}^s) = \min_{x \in D} F_q(x; x_{q-1}^s)$$

$$F_q(x; x_{q-1}^s) = \sum_{i=1}^{l} \frac{1}{f_i^s - f_i(x) + \xi_i^q} + \sum_{i=1}^{m} \frac{1}{g_i(x) + \varepsilon}.$$

Here $\varepsilon > 0$ is the sufficiently small given value,

$$\xi_i^q = \begin{cases} f_i(x_q^s) - f_i^s + \varepsilon & f_i(x_q^s) \ge f_{i_s}^s, \\ 0 & f_i(x_q^s) < f_i \end{cases}.$$

An algorithm for consistent implementation of criterion truncations is used. Correlation properties of the objective functions $f_i(x), i = 1,...,l$ and constraint functions $g_j(x), j = 1,..,m$ are used in the developed algorithm. For this purpose, an extended matrix $F(x) = (f_1(x),...,f_l(x), g_1(x),...,g_m(x))$ and symmetric $(m+n)$ dimensional matrix are entered. Elements of the matrices are:

$$r_{ij}(x) = (\nabla F_i(x), \nabla F_j(x)) / \|\nabla F_i(x)\| \cdot \|\nabla F_j(x)\|),$$

$$i, j = 1,...,m+n,$$

where $r_{ij} \in [-1;1]$. Elements of the matrices characterize the degree of antagonism (inconsistency) of functions at the current point $x^s \in D$. Depeds on the valuis of the $r_{ij}(x^s)$ and proximity of x^s to the boundaries of D, i.e. to $g_j(x) = 0, \quad j = 1,...,m$ new desirable values of the criteria $f_i(x), i = 1,...,l$ are assigned.

A subsystem, which implements the method of UDS-searching allows you to explore a given area, using the points generated by a number of UDS, which possess the best characteristics of uniformity among all currently known uniformly distributed sequences. In each of these points $P_1,...,P_N$ values of all criteria are calculated, and $F_v^* = \min F_v(P_i), i = 1,...,N$ is considered the approach to the lowest value of any criteria. The user selects a final decision from the calculated set, based on the specifics of the problem, or any additional considerations that it was not reflected in the formulation of the problem. The advantage of this approach is that the time-consuming calculations are performed only once. A sorting of the points of the given area of parameter space is carried out by taking into account the functional limitations. Also it is realized a possibility of taking into account criterion constraints that are specified in the process of dialog. This give us a possibility to limit the number of solutions that have been selected. The subsystem provides the decision-makers these choices , and he selects one of them as an optimal solution. The subsystem provides the ability through dialogue with DM to focus on some points for further testing of localized areas.

A subsystem, which implements the method of constructing the Pareto set by using Peano type curves allow to generate the one-dimentional function of convolution and use the properties of parametrical problems of the mathematical programming [2]. The essence of this method is as follows. We consider the finite-dimentional problem of vector optimization:

$$(f_1(x),...,f_l(x) \to \min_{x \in D}, \quad (4)$$

$$D = \{x \in E^n : f_i(x) = 0, i = l+1,...,m\}, \quad (5)$$

where $f_i(x), i = 1,...,l$ are convex twice- differentiable functions. In the problem (4)-(5) the Pareto set of efficient (not be improved) points

$$\Pi_x = \{X \in D : \bar{\exists}\, y \in D, f_i(y) \le f_i(x), i = 1,...,l\} \quad (6)$$

is essential. This approach involves the construction ε - covering of the Pareto set (ε is required accuracy of the solution). Function of criteria convolution is:

$$F(x, \lambda^p) = \sum_{i=1}^{l} \lambda_i f_i(x), \quad \lambda^p = (\lambda_1,...,\lambda_l) \in \Lambda, \quad (7)$$

$$\Lambda = \left\{ \lambda^p : \sum_{i=1}^{l} \lambda_i = 1, \ \lambda_i \ge 0, i = 1,...,l \right\}. \quad (8)$$

It is known that the point

$$x^*(\lambda^p) = \arg\min_{x \in D} F(x, \lambda^p) \quad (9)$$

for every fixed $\lambda^p \in \Lambda$ belongs to Pareto set and inversely, for every point of Pareto set $x \in \Pi_x$ there is some vector $\lambda^p \in \Lambda$ satisfied (9). For the other side, (9) is a parametrical problem of the constrained optimization, in which the area of the parameters is fixed by correlations (8). The Lagrangian

$$L(x,\lambda)=\sum_{i=1}^{l}\lambda_i f_i(x)+\sum_{j=l+1}^{m}\lambda_j^L f_j(x),$$

where $\lambda^L=(\lambda_{l+1},...,\lambda_m)$ is the vector of the Lagrangian dual variables, is introduce. $\lambda=(\lambda^P,\lambda^L)$. The necessary condition of optimality of vector x_{λ^P} in the problem (9), corresponding to vector λ^P is the existence of vector λ^L for which the following takes place:

$$\nabla L(x_\lambda,L)=\sum_{i=1}^{l}\lambda_i\nabla f_i(x_\lambda)+\sum_{j=1+l}^{m}\lambda_j\nabla f_j(x_\lambda)=0. \quad (10)$$

Let x_λ is a solution of the problem (9) for given vector $\lambda^{P^0}=(\lambda_1^0,...,\lambda_l^0);$ and $\lambda^{L^0}=(\lambda_{l+1}^0,...,\lambda_m^0)$ is the corresponding vector of the dual variables; $\lambda^0=(\lambda_1^0,...,\lambda_m^0)$. Let solution $x=x_\lambda+\Delta x$ corresponds to vector $\lambda^P=\lambda^{P^0}+\Delta\lambda^P$ from (9), where $\Delta\lambda^P$ is the arbitrary increment. From necessary conditions of optimality at the point $(x+\Delta x,\lambda^P+\Delta\lambda)$, using decomposition of the function at (10) to the members of the first order, it is not difficult to obtain correlations:

$$\sum_{i=1}^{l}\lambda_i^0 H_i(x_{\lambda^0},\lambda^0)\Delta x+\sum_{j=1+j}^{m}\lambda_j^0 Hj(x_{\lambda^0},\lambda^0)\Delta x+ \\ +\sum_{j=1+j}^{m}\Delta\lambda_j^0\nabla f_j(x_{\lambda^0})=\sum_{i=1}^{l}\Delta\lambda_i\nabla f(x_{\lambda^0}), \quad (11)$$

$$f_j(x)+\nabla f_j(x)\Delta x=0,\ \ j=l+1,...,m, \quad (12)$$

where matrix $H_i(x,\lambda)=((\frac{\partial^2 f_i}{\partial x_j\partial x_k})),\ j,k=1,...,n$ are the Hessians of corresponding functions, $i=1,...,m$. (11),(12) is the system of linear equations of the order $n+(m-l)$ concerning Δx and $\Delta\lambda^L$, for its solution one of the corresponding numerical method is used. Thus, for obtaining every new point of Pareto set, corresponding to new values of coefficients (parameters) of the convolution function, it is necessary to solve the system of linear equations (11), (12). It is much more easily than solving the optimization problem (9). In the process of obtaining efficient points it is necessary to carry out a correction (returning into Pareto set) because of approximation errors, which had place in (12). In this case a value of criteria of exit of Π_x of a current point, which was obtained by solving the system (11),(12), is a discrepancy by relations (10). If is necessary to carry out a correction (i.e. if the discrepancy is quite substantial), the "additional" optimization of the problem (9) is carried out. It uses one of the numerical methods. In this case the matrix $H_i(x)$, calculated at previous point., might be used. For regulating and systematizing of the obtaining points it is necessary to probe a set Λ so that the transition from one point to another was carried out continuously. In this case we use an algorithm of one-dimensional development of multi-dimensional convolution function using Peano type curves. The algorithm is implemented in such a way:

1. for the initial λ^0 we define $x_{\lambda^0}=\arg\min_{x\in D}F(x)$ and $\lambda_j, j=j+1,...,m$ from the condition $\nabla F(x_\lambda,\lambda_0)=-\sum_{j=1+l}^{m}\lambda_j\nabla f_j(x_\lambda)$;
2. we take a step along the Peano curve: $\lambda^P=\lambda_0^P+\Delta\lambda)$;
3. we solve the system of the equations (11, (12) and define the vectors $x_\lambda,\lambda_j, j=1+j,...,m$;
4. if it is necessary, the correction of solution, obtained by "additional" optimization of problem (9) is carried out of a current point.

Another approach that provides a continuous method of constructing Pareto sets, allows to generate all the efficient points, which form the Pareto-optimal surface of the problem[3]. This allows on the next stage of the decision making to provide the DM with whole Pareto set of solutions at once. At the same time if the DM wishes, there is a possibility to compare decisions that were provided him. It is suggested that the DM has an implicit (not formalized) function of value $F(f_1,f_2,...,f_n)$, with respect to which the solution will be accepted. The point $x^0\in\Omega$ is called the efficient point (not be improved) in Ω with respect to $f(x)$, if there is no other such a point $\bar{x}\in\Omega$ that $f(\bar{x})\le f(x^0)$ is true. If x^0 is an efficient point, then there exists a vector $\lambda=(\lambda_1,...,\lambda_l)$ with components $\lambda_i>0$ satisfying the equation $\sum_{i=1}^{l}\lambda_i=1$, that the maximum of the function $Z(x,\lambda)$ of the set of x satisfying the constraints $g(x)\ge 0,\ \ x\in\Omega$ is achieved for all $x=x^0$. It is base of the method of generating the points of the Pareto set, which consists in optimizing functions $Z(x,\lambda)$ for different λ, what is equivalent to a linear convolution of criteria.

To construct the set of efficient points the method based on the assertion of the fundamental Poincare recurrence theorem relatively closed-loop physical system is used. The theorem implies that a material closed -loop system should, if it is not in special condition, to pass in the end, arbitrarily close to all points of the surface of constant energy. It is assumed that the surface Ω $2n$ -dimensional space is compact and quite regular. The trajectory of the closed-loop system is entirely on the Pareto set of vector optimization problem and covers it, what means that each point of the trajectory is the efficient point. Describing the motion of a closed-loop physical system by a system of differential equations and solving it with the initial point belonging to the Pareto set (for that we need to solved the problem of finding the minimum of the convolution only once), we find a set of solutions also belonging to the Pareto set.

In the process of dialogue, the system of vector optimization makes it possible to obtain sufficient information about the interdependence of both conflicting and agreed criteria. This information allows decision-makers the implementation of guaranteed choice from a variety of best possible solutions. The system is implemented in Visual Basic in Windows and is designed for a wide user base.

IV. Applications of Interactive System of Vector Optimization

Vector optimization problems are found in widely varying areas of human activity. One of the important applications is optimization in the technical design field. For example, the task of designing a computer with a maximum speed, maximum memory and minimum weight. Problems arising during the extraction, processing and transportation of liquid and gas are also multi-criteria problems [4]. The developed interactive system can be successfully applied to solve the problem of operational planning activities of the enterprise. The developed interactive system was tested on some problems of optimal control, but the adaptation of the system for specific class of problems required some setup software.

Operating the system requires a strategy by the user in the sense of using any specific tasks. The strategy of user is mainly determined by his knowledge of the problem, the availability of information about the behavior of the objective functions, and constrained functional. At the same time, characteristics such as smoothness and continuity of the functions and their derivatives, the ranges of scalar criteria, the behavior of functions in a particular sub-area of the feasible set, position constraints, the knowledge of good initial approximation are important.

References

[1] K.R. Ayda-zade, M.N. Khoroshko, "Method of criteria truncation for nonlinear problems of vector optimization", Transactions of Azerbaijan National Academy of Sciences, series of physics-technical and mathematical sciences, №5, Baku, Azerbaijan, 1988 (in Russian).

[2] K.R. Ayda-zade, M.N. Khoroshko, " On a method of covering a Pareto set of multi-criteria optimization problem", Transactions of Azerbaijan National Academy of Sciences, series of physics-technical and mathematical sciences, №4, Baku, Azerbaijan, 1994 (in Russian).

[3] K.R. Ayda-zade, M.N. Khoroshko, "Parametrical Method for the construction of the Pareto Set of the multi-criterion Optimization Problem". Proceeding of the Second International Symposium on Mathematical and Computational Applications, Baku, Azerbaijan, 1995.

[4] Sadykhov R.A., Khoroshko M.N., "Multi-criteria problems of oil extraction and construction a Pareto set", News of Azerbaijan High Technical Educational Institutions, Baku, Azerbaijan, №4, 1994 (in Russian).

[5] Mattias Ehrgot, "Multi-criteria Optimization", Second Edition, Springer, 2005, [pp.16, 65,80]

[6] Mueller-Gritschneder, Daniel; Graeb, Helmut; Schlichtmann, "A Successive Approach to Compute the Bounded Pareto Front of Practical Multi-objective Optimization Problems". SIAM Journal on Optimization, 2009, [pp. 915–934].