Necessary Optimality Conditions of the First and Second Orders in the Problem of Control of Processes Described by Difference Analogy of Volterra Equation under Equality and Inequality Type Functional Constraints

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***Abstract*— A discrete problem of optimal control is considered. Necessary optimality conditions are obtained.**

Key words*— *first order necessary conditions; difference analogy; optimal control; equation in variations

I. Introduction

We study an optimal control problem described by a system of Volterra difference equations (see [1]) involving equality and inequality type constraints on the right end of the trajectory. Necessary optimality conditions of first and second orders are obtained. Therewith, the scheme suggested by the author in the papers [2, 3] is used.

II. Statement of Problem

Consider a problem on minimum of the terminal functional

$$S_0(u)=\varphi_0(x(t_1)), \tag{1}$$

under the constrains

$$u(t)\in U\subset R^r,\ \ t\in T=\{t_0,t_0+1,\ldots,t_1\}, \tag{2}$$

$$\begin{aligned} &S_i(u)=\varphi_i(x(t_1))\le 0, \quad i=\overline{1,p},\\ &S_i(u)=\varphi_i(x(t_1))=0, \quad i=\overline{p+1,q}, \end{aligned} \tag{3}$$

$$x(t)=\sum_{\tau=t_0}^{t} f(t,\tau,x(\tau),u(\tau)),\ \ t\in T. \tag{4}$$

Here t_0,t_1 are the given numbers, moreover the difference t_1-t_0 is a natural number, $\varphi_i(x)$, $i=\overline{1,q}$ are twice continuously differentiable scalar functions, U is a given non-empty, bounded and open set, $u(t)$ is an r-dimensional vector of control actions, $f(t,\tau,x,u)$ is a given n-dimensional vector-function continuous in totality of variables together with partial derivatives with respect to (x,u) to second order inclusively.

The control function $u(t)\in U$, $t\in T$ is called an admissible control, if the solution $x(t)$ of equation (4) corresponding to it satisfies constraint (3).

III. Necessary Optimality Conditions

Assume that $(u(t),x(t))$ is a fixed admissible process and introduce the denotation

$$\begin{aligned} &H(t,x(t),u(t),\psi_i(t))=\\ &=-\frac{\varphi_i'(x(t_1))}{\partial x} f(t_1,t,x(t),u(t))+\\ &+\sum_{\tau=t}^{t_1}\psi_i'(\tau)f(\tau,t,x(t),u(t)),\\ &H_u^{(i)}(t)\equiv H_u(t,x(t),u(t),\psi_i(t)),\\ &H_x^{(i)}(t)\equiv H_x(t,x(t),u(t),\psi_i(t)),\\ &H_{uu}^{(i)}(t)\equiv H_{uu}(t,x(t),u(t),\psi_i(t)),\\ &H_{xx}^{(i)}(t)\equiv H_{xx}(t,x(t),u(t),\psi_i(t)),\\ &Q(t,\tau)=f_u(t,\tau,x(\tau),u(\tau))-\\ &-\sum_{s=\tau}^{t}R(t,s)f_u(s,\tau,x(\tau),u(\tau)), \end{aligned}$$

$$\begin{aligned} K_i(\tau,s)&=-Q'(t_1,\tau)\frac{\partial^2\varphi_i(x(t_1))}{\partial x^2}Q(t_1,s)+\\ &+\sum_{t=\max(\tau,s)}^{t_1}Q'(t,\tau)H_{xx}^{(i)}(t)Q(t,s) \end{aligned} \tag{5}$$

and equation in variations

$$\delta x(t) = \sum_{\tau=t_0}^{t} \left[f_x(t,\tau,x(\tau),u(\tau))\delta x(\tau) + \right. \\ \left. + f_u(t,\tau,x(\tau),u(\tau))\delta u(\tau) \right]. \tag{6}$$

Here $\delta u(t) \in R^r$, $t \in T$ is an arbitrary bounded vector-function (admissible control variation), $R(\tau,t)$ is an $(n \times n)$ matrix function (resolvent of the variational equation (6) for system (4)) being a solution of the matrix difference equation of Volterra type

$$R(\tau,t) = \sum_{s=t}^{\tau} R(\tau,s) f_x(s,t,x(t),u(t)) - \\ - f_x(\tau,t,x(t),u(t)). \tag{7}$$

Assume that $(u(t),x(t))$ is an optimal process in the considered problem and for all $i = \overline{1,p}$, $S_i(u(t)) = 0$.

The first and second variations of the functional have the forms:

$$\delta^1 S_i(u;\delta u) = -\sum_{t=t_0}^{t_1} H_u^{(i)}(t)\delta u(t), \tag{8}$$

$$\delta^2 S_i(u;\delta u) = \delta x'(t_1) \frac{\partial^2 \varphi_i(x(t_1))}{\partial x^2} \delta x(t_1) - \\ - \sum_{t=t_0}^{t_1} \left[\delta x'(t) H_{xx}^{(i)}(t) \delta x(t) +. \right. \\ \left. + 2\delta u'(t) H_{ux}^{(i)}(t)\delta u(t) + \delta u'(t) H_{uu}^{(i)}(t)\delta u(t) \right], \tag{9}$$

Theorem 1. For optimality of the admissible control $u(t)$ in the considered problem it is necessary the existence of the vector

$$\lambda = (\lambda_0, \lambda_1, ..., \lambda_q) \in R^{q+1},\ \lambda_i \geq 0,\ i = \overline{0,p},$$

$$\|\lambda\| = \sum_{i=0}^{q} |\lambda_i| = 1,$$

such, that

$$\sum_{i=0}^{q} \lambda_i \delta^1 S_i(u : \delta u) = 0,$$

for all $\delta u(t) \in R^r$, $t \in T$.

Each admissible control $u(t)$ satisfying the conditions theorem 1, is called classic extremals.

Following the scheme suggested in [2, 3], the second variation (9) of the functional $S_i(u)$ is represented in the form

$$\delta^2 S_i(u;\delta u) = -\left[\sum_{t=t_0}^{t_1} \delta u'(t) H_{uu}^{(i)}(t) \delta u(t) + \right. \\ + \sum_{\tau=t_0}^{t_1} \sum_{s=t_0}^{t_1} \delta u'(\tau) K_i(\tau,s) \delta u(s) + \\ \left. + 2 \sum_{t=t_0}^{t_1} \left[\sum_{\tau=t_0}^{t_1} \delta u'(t) H_{ux}^{(i)}(t) Q(t,\tau) \delta u(\tau) \right] \right].$$

Denote by $A(u)$ the set of all vectors $\lambda \in R^{q+1}$, satisfying the conditions of theorem 1, and introduce the analogy of the [4, 5] set of critical variations for the considered problem.

$$DM(u;\delta u) = \{ \delta u : \delta^1 S_i(u;\delta u) \leq 0, i = \overline{0,p}; \\ \delta^1 S_i(u;\delta u) = 0, i = \overline{p+1,q} \}.$$

Theorem 2. For optimality of classical extremals in problem (1)-(4) it is necessary that the inequality

$$\max_{\lambda \in A(u)} \sum_{i=0}^{q} \lambda_i \delta^2 S_i(u;\delta u) \geq 0 \tag{10}$$

be fulfilled for all $\delta u(t) \in DM(u;\delta u)$.

IV. Conclusion

An optimal control problem described by a system of Volterra type difference equations involving equality and inequality type functional constraints on the right end of the trajectory, is studied. Necessary optimality conditions of first and second orders are established.

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