Problem of Optimal Control of Moving Sources for Systems with Distributed Parameters

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***Abstract*— A problem on optimal control of processes described by totality of a parabolic type equation and ordinary differential equation with moving sources is investigated in the paper. In spite of applied importance of problems with moving sources controls, at present they have not been studied enough. A theorem on existence and uniqueness of the solution is solved for the optimal control, sufficients conditions of Frechet differentiability of quality test and an expression for its gradient are obtained, necessary conditions of optimality in the form of point wise and integral maximum principles are established for an optimal control problem considered in the work.**

***Keywords*— moving sources; reduced problem; necessary conditions of optimality; maximum principles**

I INTRODUCTION

Let's consider controlled process described by the parabolic equation

$$u_t = a^2 u_{xx} + \sum_{k=1}^{n} p_k(t)\delta(x - s_k(t)),\ (x,t) \in \Omega, \quad (1)$$

with boundary and initial conditions

$$u_x\big|_{x=0} = g_1(t),\ u_x\big|_{x=l} = g_2(t),\ 0 < t \le T, \quad (2)$$

$$u(x,o) = \varphi(x),\ 0 \le x \le l, \quad (3)$$

where $a, l, T > 0$ are given numbers; $\delta(\cdot)$ - Dirac's function; $\Omega = \{(x,t) : 0 < x < l, 0 < t \le T\}$; $\varphi(x) \in L_2(0,l)$, $g_i(t) \in L_2(0,T)$ $(i = \overline{1,2})$ -are the known functions; $p(t) = (p_1(t), p_2(t),\ldots,p_n(t)) \in L_2^n(0,T)$- is the control function. The functions $s_k(t) \in H_1(0,T)(k = \overline{1,n})$ are supposed the solutions of the following Cauchy problem

$$\begin{aligned} &\dot{s}_k(t) = f_k(s_k(t), \vartheta(t), t),\ 0 < t \le T, \\ &s_k(o) = s_{k0},\ k = \overline{1,n}, \end{aligned} \quad (4)$$

where s_{k0}-are the given real numbers; $\vartheta = \vartheta(t) = (\vartheta_1(t), \vartheta_2(t),\ldots,\vartheta_r(t)) \in L_2^r(0,T)$ - is the control function; functions $f_k(s_k, \vartheta, t)(k = \overline{1,n})$ is continuous, are continuous derivatives at s and ϑ for $(s,\vartheta,t) \in E^n \times E^r \times [o,T]$ $(s,\vartheta,t) \in E^n \times E^r \times [o,T]$ and they are bounded: $|f_{k\vartheta}(s,\vartheta,t)| \le M_\vartheta, |f_{ks}(s,\vartheta,t)| \le M_s, M_\vartheta > 0, M_s > 0.$

The functional spaces $W_2^{1,1}(\Omega)$, $V_2^{1,0}(\Omega)$, $H_1(0,T)$ used below are introduced, for example in [3].

Pair of functions $\overline{\vartheta} = (p(t), \vartheta(t))$ we will call controls. For brevity we denote by $H = L_2^n(0,T) \times L_2^r(0,T)$ a Hilbert space of pairs $\overline{\vartheta} = (p(t), \vartheta(t))$ with scalar product

$$< \overline{\vartheta}^1, \overline{\vartheta}^2 >_H = \int_0^T [p^1(t)p^2(t) + \vartheta^1(t)\vartheta^2(t)]dt$$

and the norm $\|\overline{\vartheta}\|_H = \sqrt{(\|p\|_{L_2}^2 + \|\vartheta\|_{L_2}^2)}$, where $\overline{\vartheta}^k = (p^k, \vartheta^k)(k = \overline{1,2})$.

Suppose, that control $\overline{\vartheta} = (p(t), \vartheta(t))$ is found on the set of admissible controls

$$V = \{(p,\vartheta) \in H : 0 \le p_i \le A_i, 0 \le \vartheta_j \le B_j,\ i = \overline{1,n}, j = \overline{1,r}\}. \quad (5)$$

Let's consider the problem of minimization of functional

$$\begin{aligned} J(\overline{\vartheta}) = &\int_0^l \int_0^T [u(x,t) - \widetilde{u}(x,t)]^2 dxdt + \\ &+ \alpha_1 \sum_{k=1}^{n} \int_0^T [p_k(t) - \widetilde{p}_k(t)]^2 dt + \\ &+ \alpha_2 \sum_{m=1}^{r} \int_0^T [\vartheta_m(t) - \widetilde{\vartheta}_m(t)]^2 dt, \end{aligned} \quad (6)$$

at the solutions $(u(x,t), s(t)) = (u(x,t;\overline{\vartheta}), s(t;\vartheta))$ of boundary value problem (1)-(4), which correspond to all admissible controls $\overline{\vartheta} \in V$, where $\alpha_1, \alpha_2 \ge 0, \alpha_1 + \alpha_2 > 0, A_i > 0(i = \overline{1,n})$, $B_j > 0(j = \overline{1,r})$ – are given numbers;

$\tilde{u}(x,t)) \in L_2(\Omega)$, $\omega = (\tilde{p}(t), \tilde{\vartheta}(t)) \in H$, $\tilde{p}(t) = (\tilde{p}_1(t), \tilde{p}_2(t), ..., \tilde{p}_n(t)) \in L_2^n(0,T)$, $\tilde{\vartheta}(t) = (\tilde{\vartheta}_1(t), \tilde{\vartheta}_2(t), ..., \tilde{\vartheta}_r(t) \in L_2^r(0,T))$ - are the known functions.

II CORRECTNESS OF PROBLEM STATEMENT

Definition. The problem about finding of functions $(u(x,t), s(t)) = (u(x,t;\bar{\vartheta}), s(t;\vartheta))$ from conditions (1)-(4) at the given control $\bar{\vartheta} \in V$ we said the reduced problem. The solution of the reduced problem (1)-(4), corresponding to the control $\bar{\vartheta} = (p(t), \vartheta(t)) \in V$, is understood as functions $(u(x,t), s(t))$ from $(V_2^{1,0}(\Omega), H_1^n(0,T))$, where the function $u = u(x,t)$ satisfying the integrated identity

$$\int_0^l \int_0^T [-u\eta_t + a^2 u_x \eta_x] dxdt = \int_0^l \varphi(x)\eta(x,0)dx + \sum_{k=1}^n \int_0^T p_k(t)\eta(s_k(t),t)dt, \tag{7}$$

for $\forall \eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ and $\eta(x,T) = 0$, where function $s_k(t) = s_k(t;\vartheta)$ satisfies the integrated equation

$$s_k(t) = \int_0^t f_k(s_k(\tau), \vartheta(\tau), \tau)d\tau + s_{k0}, \quad k = \overline{1,n},\ 0 \le t \le T. \tag{8}$$

It follows from the results of the papers [3-4] that for each $\bar{\vartheta} \in V$, the reduced problem (1)-(4) has the unique solution in $(V_2^{1,0}(\Omega), H_1^n(0,T))$. Let the conditions accepted in the statement of problem (1)-(6) be fulfilled. Then problem (1)-(6) has at least one solution. It should be noted that the problem (1)-(6) for $\alpha_j = 0 (j = \overline{1,2})$ is ill-posed in the classic sense [5]. However it holds

Theorem 1. There exists a dense subset K of the space H, such that for each $\omega \in K$ for $\alpha_i > 0 (i = \overline{1,2})$ the problem (1)-(6) has a unique solution.

III NECESSARY CONDITIONS OF OPTIMALITY

Let $\psi = \psi(x,t)$ be a solution of the problem in $V_2^{1,0}(\Omega)$:

$$\psi_t + a^2\psi_{xx} = -2[u(x,t) - \tilde{u}(x,t)],\ (x,t) \in \Omega, \tag{9}$$

$$\psi_x\big|_{x=0} = \psi_x\big|_{x=l} = 0,\ 0 \le t < T, \tag{10}$$

$$\psi(x,T) = 0,\ \ 0 \le x \le l, \tag{11}$$

which conjugated to (1)-(6), where $u(x,T)$ - value is a solution of reduced problem (1)-(4) for $t = T$ and $q_k(t)$ is a solution of conjugated problem in $H_1(0,T)$:

$$\dot{q}_k(t) = -\frac{\partial f_k}{\partial s_k} q_k(t) + p_k(t)\psi_x(s_k(t),t), \quad 0 \le t < T, q_k(T) = 0, k = \overline{1,n}. \tag{12}$$

The function $\psi = \psi(x,t)$ satisfies the i identity

$$\int_0^l \int_0^T [\psi\eta_{1t} + a^2\psi_x\eta_{1x}]dxdt = 2\int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]\eta_1(x,t)dxdt, \tag{13}$$

for $\forall \eta_1 = \eta_1(x,t) \in W_2^{1,1}(\Omega)$ and $\eta_1(x,0) = 0$, the function $q = q(t)$ satisfies the equation

$$q_k(t) = \int_t^T \left[\frac{\partial f_k}{\partial s_k} q_k(\tau) - p_k(\tau)\psi_x(s_k(\tau),\tau)\right] d\tau, \quad 0 \le t \le T, k = \overline{1,n} \tag{14}$$

The conjugated problem (9)-(12) is a mixed problem for the linear parabolic equation. If in relations (9)-(12), instead of a variable t we take a new independent variable $\tau = T - t$, we get a boundary value problem of the same type as (1)-(4).

Therefore, it follows from the facts established for problem (1)-(4) that for each given $\bar{\vartheta} = (p(t), \vartheta(t)) \in V$ problem (9)-(12) has a unique solution in $(V_2^{1,0}(\Omega), H_1^n(0,T))$.

The function

$$H(t,s,\psi,q,\bar{\vartheta}) = -\left\{\sum_{k=1}^n \Big[- f_k(s_k(t),\vartheta(t),t)q_k(t) + \psi(s_k(t),t)p_k(t) + \alpha_1\big(p_k(t) - \tilde{p}_k(t)\big)^2\Big] + \alpha_2 \sum_{m=1}^r \big(\vartheta_m(t) - \tilde{\vartheta}_m(t)\big)^2\right\}$$

is said to be Hamilton-Portraying function for problem (1)-(6).

Theorem 2. Let the functions $f_k(s,\vartheta,t) (k = \overline{1,n})$ be continuous in totality of all its arguments together with all its partial derivatives with respect to variables s, ϑ at $(s,\vartheta,t) \in E^n \times E^r \times [0,T]$ and besides, are satisfied following conditions :

$$|f_k(s + \Delta s, \vartheta + \Delta\vartheta, t) - f_k(s,\vartheta,t)| \le L(|\Delta s| + |\Delta\vartheta|),$$

$$|f_{ks}(s + \Delta s, \vartheta + \Delta\vartheta, t) - f_{ks}(s,\vartheta,t)| \le L(|\Delta s| + |\Delta\vartheta|),$$

$\left|f_{k\vartheta}(s+\Delta s,\vartheta+\Delta\vartheta,t)-f_{k\vartheta}(s,\vartheta,t)\right|\le L(|\Delta s|+|\Delta\vartheta|)$,
for every $(s+\Delta s,\vartheta+\Delta\vartheta,t)$,
$(s,\vartheta,t)\in E^n\times E^r\times[0,T]$, where $L=const\ge 0$.

Then the functional (1) is Frechet differentiable and the expression

$$J'(\overline{\vartheta})=-\frac{\partial H}{\partial\overline{\vartheta}}\equiv(-\frac{\partial H}{\partial p},-\frac{\partial H}{\partial\vartheta}),$$

where

$$\frac{\partial H}{\partial p}=\left(\frac{\partial H}{\partial p_1},\frac{\partial H}{\partial p_2},\dots,\frac{\partial H}{\partial p_n}\right),$$

$$\frac{\partial H}{\partial\vartheta}=\left(\frac{\partial H}{\partial\vartheta_1},\frac{\partial H}{\partial\vartheta_2},\dots,\frac{\partial H}{\partial\vartheta_r}\right),$$

$$\frac{\partial H}{\partial p_k}=-\psi(s_k(t),t)-2\alpha_1\left(p_k(t)-\widetilde{p}_k(t)\right),k=\overline{1,n},$$

$$\frac{\partial H}{\partial\vartheta_m}=\sum_{k=1}^{n}\frac{\partial f_k(s_k(t),\vartheta(t),t)}{\partial\vartheta_m}q_k(t)-$$
$$-2\alpha_2\left(\vartheta_m(t)-\widetilde{\vartheta}_m(t)\right),m=\overline{1,r},$$

is valid for its gradient.

Theorem 3. Let all conditions of the theorem 2 be fulfilled and $(u^*(x,t),s^*(t)),(\psi^*(x,t),q^*(t))$ be solutions of problems (1)-(4) and (9)-(12), respectively, for $\overline{\vartheta}=\overline{\vartheta}^*\in V$. Then for optimality of the control $\overline{\vartheta}^*=(p^*(t),\vartheta^*(t))$ the condition

$$H(t,s^*(t),\psi^*(x,t),q^*(t),\overline{\vartheta}^*(t))=$$
$$=\max_{\overline{\vartheta}\in V}H(t,s^*(t),\psi^*(x,t),q^*(t),\overline{\vartheta}),$$

should be fulfilled for $\forall(x,t)\in\Omega$.

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