Existence of the Optimal Trajectories of the Controllable System Described by an Affine Integral Equation

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***Abstract*— In this paper the control system with integral constraint on the controls is studied. It is assumed that the behavior of the system is described by an affine Volterra integral equation which is nonlinear with respect to the state vector and is linear with respect to the control vector. The closed ball of the space L_p $(p>1)$ with gives radius μ_0 and centered at the origin, is chosen as the set of admissible control functions. Compactness of the set of trajectories and existence of the optimal trajectories in the optimal control problems with semi-continuous cost functionals are proved.**

Keywords*— *affine integral equation; controllable system; integral constraint; set of trajectories; optimal trajectory

I. INTRODUCTION

Control systems with integral constraint on the controls arise in various fields of the theory and applications of the applied mathematics. (see., e.g. [1-5]). For example, the motion of flying objects with variable mass, is described in the form of controllable system, where the control function has an integral constraint (see., e.g. [3-5]). Control system with integral constraint on the controls whose behavior is described by a differential equation studied in [1-8]. In [6-8] various topological properties and numerical construction methods of the set of trajectories and the attainable sets of the control system with integral constraint on the controls are studied where the behavior of the system is described by an affine differential equation. Integral equations appear in many problems of contemporary physics and mechanics (see., e.g. [9-14]). In this paper the control system with integral constraint on the controls whose behavior is described by a Volterra integral equation is considered. It is assumed that integral equation is affine, i.e. it is nonlinear with respect to the state vector and is linear with respect to the control vector. The closed ball of the space L_p $(p>1)$ with radius μ_0 and centered at the origin, is chosen as the set of admissible control functions. It is proved that the set of trajectories is a compact subset of the space of continuous functions. Applying this result it is shown that optimal control problem with semi-continuous cost functional has an optimal trajectory.

Consider controllable system the behavior of which is described by an integral equation

$$x(t)=g(t,x(t))+\lambda\int_{t_0}^{t}\left[K_1(t,s,x(s))+K_2(t,s,x(s))u(s)\right]ds, \quad (1)$$

where $x\in R^n$ is the state vector, $u\in R^m$ is the control vector, $t\in[t_0,\vartheta]$. The system (1) is linear with respect to the control vector, but it is nonlinear with respect to the state vector. Therefore the system (1) will be called an affine control system.

Let $p>1$ and $\mu_0>0$ be given numbers. The function $u(\cdot)\in L_p\left([t_0,\vartheta];R^m\right)$ such that $\|u(\cdot)\|_p\le\mu_0$ is said to be an admissible control function where

$$\|u(\cdot)\|_p=\left(\int_{t_0}^{\vartheta}\|u(t)\|^p dt\right)^{\frac{1}{p}}.$$

The set of all admissible control functions is denoted by symbol U_p. Thus

$$U_p=\left\{u(\cdot)\in L_p\left([t_0,\vartheta];R^m\right):\|u(\cdot)\|_p\le\mu_0\right\}.$$

It is obvious that the set of admissible control functions is the closed ball with radius $\mu_0>0$ and centered at the origin in the space $L_p\left([t_0,\vartheta];R^m\right)$.

It is assumed that the functions and a number λ given in system (1) satisfy the following conditions:

A. the functions

$$g(\cdot,\cdot):[t_0,\vartheta]\times R^n\to R^n,$$

$$K_1(\cdot,\cdot,\cdot):[t_0,\vartheta]\times[t_0,\vartheta]\times R^n\to R^n,$$

$$K_2(\cdot,\cdot,\cdot):[t_0,\vartheta]\times[t_0,\vartheta]\times R^n\to R^{n\times m}$$

are continuous;

B. there exist $L_0\in[0,1)$, $L_1\ge 0$ *and* $L_2\ge 0$ *such that*

$$\|g(t,x_1)-g(t,x_2)\|\le L_0\|x_1-x_2\|,$$

$$\|K_1(t,s,x_1)-K_1(t,s,x_2)\| \le L_1\|x_1-x_2\|,$$
$$\|K_2(t,s,x_1)-K_2(t,s,x_2)\| \le L_2\|x_1-x_2\|$$

for every $(t,s,x_1)\in[t_0,\vartheta]\times[t_0,\vartheta]\times R^n$, $(t,s,x_2)\in[t_0,\vartheta]\times[t_0,\vartheta]\times R^n$;

C. $0\le\lambda\left(L_1(\vartheta-t_0)+L_2(\vartheta-t_0)^{\frac{p-1}{p}}\mu_0\right)<1-L_0$.

We denote

$$L(\lambda)=L_0+\lambda\left(L_1(\vartheta-t_0)+L_2(\vartheta-t_0)^{\frac{p-1}{p}}\mu_0\right). \quad (2)$$

According to the condition 2.C we obtain that $L(\lambda)<1$.

Now, let us define a trajectory of the system (1) generated by an admissible control function. Let $u_*(\cdot)\in U_p$. A continuous function $x_*(\cdot):[t_0,\vartheta]\to R^n$ satisfying the integral equation

$$x_*(t)=g(t,x_*(t))+\lambda\int_{t_0}^{t}[K_1(t,s,x_*(s))+$$
$$+K_2(t,s,x_*(s))u(s)]ds,\ t\in[t_0,\vartheta],$$

is said to be a trajectory of the system (1) generated by the admissible control function $u_*(\cdot)\in U_p$. The trajectory of the system (1) generated by the control function $u(\cdot)\in U_p$ is denoted by $x(\cdot,u(\cdot))$ and we set

$$X_p=\{x(\cdot,u(\cdot)):u(\cdot)\in U_p\}.$$

X_p is called the set of trajectories of the system (1). It is obvious that $X_p\subset C([t_0,\vartheta];R^n)$ where $C([t_0,\vartheta];R^n)$ is the space of continuous functions $x(\cdot):[t_0,\vartheta]\to R^n$ with norm

$$\|x(\cdot)\|_C=\max\{\|x(t)\|:t\in[t_0,\vartheta]\}.$$

For $t\in[t_0,\vartheta]$ we denote

$$X_p(t)=\{x(t)\in R^n:x(\cdot)\in X_p\}.$$

The set $X_p(t)$ consists of points to which arrive the trajectories of the system at the instant of t.

II. Compactness of the Set of Trajectories

Conditions A – C guarantee that every admissible control function generates a unique trajectory.

Proposition 1. *Let $u_*(\cdot)\in U_p$. Then the system (1) has unique $x_*(\cdot)=x_*(\cdot,u_*(\cdot))$ trajectory generated by the admissible control function $u_*(\cdot)$.*

Denote

$$\alpha_*=\frac{c_0+\lambda c_1(\vartheta-t_0)+\lambda c_2\mu_0(\vartheta-t_0)^{\frac{p-1}{p}}}{1-L_0},$$

$$r_*=\alpha_*\cdot\exp\left(\frac{L(\lambda)-L_0}{1-L_0}\right),$$

where L_0 is defined in condition B, $L(\lambda)$ is defined by (2),

$$c_0=\max\{\|g(t,0\|:t\in[t_0,\vartheta]\},$$
$$c_1=\max\{\|K_1(t,s,0\|:(t,s)\in[t_0,\vartheta]\times[t_0,\vartheta]\},$$
$$c_2=\max\{\|K_2(t,s,0\|:(t,s)\in[t_0,\vartheta]\times[t_0,\vartheta]\}.$$

Proposition 2. *For every $x(\cdot)\in X_p$ the inequality*

$$\|x(\cdot)\|_C\le r_*$$

holds.

The following propositions characterize the equicontinuity and closedness of the set of trajectories

Proposition 3. *The set of trajectories X_p is a set of equicontinuous functions in the space $C([t_0,\vartheta];R^n)$.*

Proposition 4. *The set of trajectories X_p is a closed subset of the space $C([t_0,\vartheta];R^n)$.*

From Propsitions 1 - 4 and Arzela – Ascoli theorem we obtain compactness of the set of trajectories.

Theorem 1. *The set of trajectories X_p is a compact subset of the space $C([t_0,\vartheta];R^n)$.*

The following theorem specifies continuity of the sections of the set of trajectories with respect to t.

Theorem 2. *The set valued map $t\to X_p(t)$, $t\in[t_0,\vartheta]$, is continuous in Hausdorff metric. i.e.*

$$h(X_p(t),X_p(t_*))\to 0 \ \ as \ \ t\to t_*$$

for every $t_*\in[t_0,\vartheta]$.

Here $h(E,G)$ denotes the Hausdorff distance between the sets $E\subset R^n$ and $G\subset R^n$ and is defined as

$$h(E,G)=\max\left\{\sup_{x\in E}d(x,G),\sup_{y\in G}d(y,E)\right\},$$

where $d(x,G)=\inf\{\|x-y\|: y\in G\}$.

III. OPTIMAL CONTROL PROBLEM

Let $\gamma(\cdot): C([t_0,\vartheta];R^n)\to R$ be a given lower semi-continuous functional. Consider following optimal control problem:

Problem A. *It is required to define* $x_*(\cdot)\in X_p$ *such that the equality*

$$\gamma(x_*(\cdot))=\inf\{\gamma(x(\cdot)): x(\cdot)\in X_p\}$$

is satisfied.

In this case $x_*(\cdot)\in X_p$ is called an optimal trajectory, the admissible control function $u_*(\cdot)\in U_p$ which generates the trajectory $x_*(\cdot)$ is said to be an optimal control function.

From compactness of the set of trajectories X_p and lower semi-continuity of the functional $\gamma(\cdot): C([t_0,\vartheta];R^n)\to R$ we have that the problem A has a solution. Thus, following theorem is valid.

Theorem 3. *The problem A has a solution, i.e. there exists* $x_*(\cdot)\in X_p$ *such that*

$$\gamma(x_*(\cdot))=\min\{\gamma(x(\cdot)): x(\cdot)\in X_p\}.$$

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