Some Approaches to Solve Feedback Control Problems for Nonlinear Dynamic Systems

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***Abstract*— Optimal feedback control problems for objects described by ordinary differential equations systems on the class of zonal control functions involving uncertain information on the values of these objects' initial conditions and parameters are investigated in the work. Zonal values of the controls are optimized in the problem considered. Formulas for the gradient of the target functional are derived. These formulas allow solving the problem numerically using efficient first order optimization methods. Results of numerical experiments carried out by the example of solution to model problems are given.**

***Keywords*— feedback control; gradient of functional; adjoint system; discrete observations; continuous observations**

I. Introduction

Feedback control problems have been investigated by many authors. Interest to this class of problems has been growing for the last decades due to development of technical, computing, and measuring tools for monitoring and controlling the state of technical objects and of technological processes. They generally considered linear systems; in the case of nonlinear systems the corresponding linearized systems were used [1-4].

In this work we investigate a class of feedback control problems for dynamic, in the general case, nonlinear objects involving lumped parameters. For synthesized control actions we introduce the notion of zonality that means constancy of the synthesized control parameters' values in each of the subsets (zones). These subsets are obtained by partitioning the set of all possible states of the object investigated. Formulas for the gradient of the target functional with respect to the optimizable parameters of the synthesized controls are obtained. These formulas can be used to formulate necessary optimality conditions and to build numerical solution schemes on basis of first order iterative optimization methods. Results of numerical experiments obtained by solving some test problems are given.

II. Problem Statement

Let the controlled process be described by the following nonlinear differential equations system:

$$\dot{x}(t) = f(x(t), u(t), p)\,,\ t > 0\,, \tag{1}$$

$$x(0) = x^0 \in X^0 \subset R^n\,,\ p \in P \subset R^m\,, \tag{2}$$

where $x(t)$ is the n-dimensional vector function of the process state; $u(t) \in U$ is the r-dimensional control vector function; $U \subset R^r$ is the closed set of the control actions' admissible values; p is the m-dimensional vector of the process's constant parameters, the values of which are uncertain, but there is a set of their possible values P and the density (weighting) function $\rho_P(p) \ge 0$ defined on P; X^0 is the set of possible values of the process's initial states with the density (weighting) function $\rho_{X^0}(x^0) \ge 0$ given.

Control of the process (1) is realized with the use of feedback; the state vector $x(t)$ may be measured fully or partially. Observations of the process state may be carried out at discrete points of time or continuously. To control the process, we propose to choose the values of the synthesized control actions according to a subset the current measured process state belongs to. The subsets are obtained by partitioning the set of all possible phase states of the object.

The objective of the feedback control problem considered is to determine the values of the parameters of the zonal control actions $u(t)$ minimizing the following functional

$$J(u) = \frac{\int\limits_{X^0}\int\limits_{P} I(u,T;x^0,p)\rho_{X^0}(x^0)\rho_P(p)dPdX^0}{(mesX^0 \cdot mesP)}\,, \tag{3}$$

$$I(u,T;x^0,p) = \int\limits_0^T g(x(t),u(t))dt + \Phi(x(T),T)\,. \tag{4}$$

Here $x(t) = x(t;x^0,p,u)$ is the solution to the system (1) under the admissible control $u(t)$, initial state x^0, and the values of the parameters p; $T = T(x^0,P)$ is the corresponding completion time of the process, which can be either a fixed quantity $T = T(x^0,P) = const = \overline{T}$, or an optimizable function of the values of the initial state and of the object's parameters

$$T = \left\{T(x^0,P) : T(x^0,P) \le \overline{T},\ x^0 \in X^0,\ p \in P\right\},$$

where $\overline{T}$ is given. The latter case arises in, as a rule, speed-in-action problems for control systems. We are to consider the both cases.

The functional (3) and (4) defines the quality of control which is optimal on the average with respect to the admissible values $x^0 \in X^0$ and $p \in P$. Denote by $X \subset R^n$ the set of all possible states of the object under different admissible initial states $x^0 \in X^0$, the values of the parameters $p \in P$, and the controls $u(t) \in U$ for $t \in [0, \overline{T}]$.

Let the set X be partitioned into given number L of open subsets X^i such that

$$\bigcup_{i=1}^{L} \overline{X}^i = X,\ X^j \cap X^i = \varnothing,\ i \neq j,\ i,j = 1,2,...,L,$$

where $\overline{X}^i$ is the closure of the set X^i.

In the work we consider the following four types of feedback control problems, which differ in organization of feedback with the object and, therefore, in formation of the control actions' values.

Problem 1

There are points of time $\tau_j \in [0,\overline{T}],\ j = 0,1,...,N,\ \tau_0 = 0$ given, at which it is possible to observe the current state of the process $x(\tau_j) \in X$. The frequency of these observations is such that when the process state belongs to some zone, it is observed at least once. The values of the control $u(t)$, which are constant for $t \in [\tau_j, \tau_{j+1})$, are assigned depending on the value of the last observed current process state, namely, depending on the subset X^i, $i = 1,2,...,L$ of the phase space X the current measured (observed) process state belongs to. Therefore

$$u(t) = v^i = const,\ x(\tau_j) \in X^i,\ t \in [\tau_j, \tau_{j+1}), \tag{5}$$

$$v^i \in U \subset R^r,\ \ i = 1,2,...L,\ \ j = 0,1,...N-1,\ \tau_N = \overline{T}.$$

It is required to determine zonal values of the control v^i, $i = 1,2,...L$, optimizing the functional (3).

Problem 2

The control actions are defined by a linear function of the results of observations of the state variables at given discrete points of time $\tau_i \in [0,\overline{T}],\ \ i = 0,1,...,N$:

$$u(t) = K_1^i \cdot x(\tau_j) + K_2^i,\ t \in [\tau_j, \tau_{j+1}),\ x(\tau_j) \in X^i,$$

$$t \in [0,\overline{T}],\ i = 1,2,...,L,\ j = 0,1,...,N-1. \tag{6}$$

Here K_1^i is the $(r \times n)$ matrix and K_2^i is the r-dimensional vector which are constant for $t \in [\tau_{j-1}, \tau_j)$. The problem is to determine the values $K_1^i,\ K_2^i,\ i = 1,2,...,L$ optimizing the functional (3).

Problem 3

Continuous observation of the process state is carried out; the control actions take zonal values of the control:

$$u(t) = w^i = const,\ x(t) \in X^i,\ t \in [0,\overline{T}],$$

$$w^i \in U \subset R^r,\ i = 1,2,...,L. \tag{7}$$

It is required to determine the zonal values of the control w^i, $i = 1,2,...L$, optimizing the functional (3).

Problem 4

Continuous observation of the process state is carried out; the control actions are defined by a linear function of the measured current values of the process variables:

$$u(t) = L_1^i \cdot x(t) + L_2^i,\ x(t) \in X^i,\ t \in [0,\overline{T}],$$

$$i = 1,2,...,L,\ j = 0,1,...,N-1. \tag{8}$$

Here L_1^i is the $(r \times n)$ matrix and L_2^i is the r-dimensional vector which are constant for each zone X^i, i.e. for $x(t) \in X^i$. It is required to determine the values $L_1^i,\ L_2^i,\ i = 1,2,...,L$, optimizing the functional (3).

Note that in all the four problems, the synthesized controls are defined by the finite-dimensional constant vectors and matrices.

III. Numerical solution to the problem

To solve the optimization problems stated above numerically and to determine the control actions $u(t),\ \ t \in [0,\overline{T}]$ from the classes (5)-(8), we propose to use first order optimization methods and the corresponding standard software [5]. As is known, to organize iterative procedures of the first order methods, one has to obtain formulas for the gradient of the target functional. Using the technique of the target functional increment obtained at the expense of the optimizable arguments increment [6], we can prove the following theorems.

Theorem 1

The components of the gradient of the target functional on the class of controls (5) are determined by the formulas:

$$\frac{\partial J(u)}{\partial v^l} = \frac{\int\limits_{X^0}\int\limits_{P} \frac{\partial I(u,T;x^0,p)}{\partial v^l} \rho_{X^0}(x^0)\rho_P(p)dPdX^0}{(mesX^0 \cdot mesP)},$$

$$\frac{\partial I(u,T;x^0,p)}{\partial v^l} = \int\limits_{\Pi_l(x^0,p,u)} \left[\frac{\partial g(x(t;x^0,p,u),u)}{\partial u} - \right.$$

$$\left. -\psi^T(t;x^0,p,u) \cdot \frac{\partial f(x(t;x^0,p,u),u,p)}{\partial u} \right] dt,$$

where $\Pi_l(x^0,p,u) = \bigcup\limits_{j: x(\tau_j;x^0,p,u) \in X^l} [\tau_j, \tau_{j+1})$, $l = 1,2,...,L$; the function $\psi(t;x^0,p,u)$ is the solution to the following adjoint Cauchy problem

$$\psi(\overline{T};x^0,p,u)=-\frac{\partial\Phi(x(\overline{T};x^0,p,u),\overline{T})}{\partial x},$$

$$\dot{\psi}^T(t;x^0,p,u)=\frac{\partial g(x(t;x^0,p,u),u)}{\partial x}-$$
$$-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial x},$$

for $t\in[0,\overline{T})$, under observance of the condition (5).

Theorem 2

The components of the gradient of the target functional on the class of controls (6) are determined by the formulas:

$$\frac{\partial J(u)}{\partial K_1^s}=\frac{1}{(mesX^0\cdot mesP)}\int_{X^0}\int_{P}\int_{\Pi_s(x^0,p,u)}\left[\frac{\partial g(x(t;x^0,p,u),u)}{\partial u}-\right.$$
$$\left.-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial u}\right]^T dt\cdot$$
$$\cdot x^T(\tau_i;x^0,p,u)\cdot\rho_P(p)\cdot\rho_{X^0}(x_0)dPdX^0,$$

$$\frac{\partial J(u)}{\partial K_2^s}=\frac{1}{(mesX^0\cdot mesP)}\int_{X^0}\int_{P}\int_{\Pi_s(x^0,p,u)}\left[\frac{\partial g(x(t;x^0,p,u),u)}{\partial u}-\right.$$
$$\left.-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial u}\right]^T dt\cdot$$
$$\cdot\rho_P(p)\cdot\rho_{X^0}(x^0)dPdX^0,$$

where $\Pi_s(x^0,p,u)=\bigcup_{j:x(\tau_j;x^0,p,u)\in X^l}[\tau_j,\tau_{j+1})$, $s=1,2,...,L$; $\psi(t;x^0,p,u)$ is the solution to the following adjoint Cauchy problem:

$$\psi(\overline{T};x^0,p,u)=-\frac{\partial\Phi(x(\overline{T};x^0,p,u),\overline{T})}{\partial x},$$

$$\dot{\psi}^T(t;x^0,p,u)=\frac{\partial g(x(t;x^0,p,u),u)}{\partial x}-\psi^T(t;x^0,p,u)\cdot$$
$$\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial x}+\sum_{s=1}^{N-1}\int_{\tau_s}^{\tau_{s+1}}\left[\frac{\partial g(x(\tau;x^0,p,u),u)}{\partial u}-\right.$$
$$\left.-\psi^T(\tau;x^0,p,u)\cdot\frac{\partial f(x(\tau;x^0,p,u),u,p)}{\partial u}\right]d\tau\cdot K_1^s\cdot\delta(t-\tau_s),$$

for $t\in[0,\overline{T})$, under observance of the condition (6).

Theorem 3

The components of the gradient of the target functional on the class of controls (7) are determined by the formulas:

$$\frac{\partial J(u)}{\partial w^k}=\frac{\int_{X^0}\int_{P}\frac{\partial I(u,T;x^0,p)}{\partial w^k}\rho_P(p)\rho_{X^0}(x^0)dPdX^0}{(mesX^0\cdot mesP)},$$

$$\frac{\partial I(u,T;x^0,p)}{\partial w^k}=\int_{\Pi_k(x^0,p,u)}\left[\frac{\partial g(x(t;x^0,p,u),u)}{\partial u}-\right.$$
$$\left.-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial u}\right]dt,$$

where

$$\Pi_k(x^0,p,u)=\{t\in[0,T]:x(t;x^0,p,u)\in X^k\},$$

$$k\in\{1,2,...,L\};\ \psi(t;x^0,p,u)$$

is the solution to the following adjoint Cauchy problem

$$\psi(\overline{T};x^0,p,u)=-\frac{\partial\Phi(x(\overline{T};x^0,p,u),\overline{T})}{\partial x},$$

$$\dot{\psi}^T(t;x^0,p,u)=\frac{\partial g(x(t;x^0,p,u),u)}{\partial x}-$$
$$-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial x},$$

for $t\in[0,\overline{T})$, under observance of the condition (7), which satisfies the following jump condition on the interface of the zones:

$$\psi^T(t_{l,m}-0;x^0,p,u)=\psi^T(t_{l,m}+0;x^0,p,u)-$$
$$-\frac{\partial h_{l,m}(x(t_{l,m};x^0,p,u))}{\partial x}\cdot\sigma_{l,m},$$

$$\sigma_{l,m}=\frac{\psi^T(t_{l,m}+0;x^0,p,u)}{\frac{\partial h_{l,m}(x(t_{l,m};x^0,p,u))}{\partial x}\cdot f(x(t_{l,m};x^0,p,u),w^l,p)}\cdot$$
$$\cdot[f(x(t_{l,m};x^0,p,u),w^l,p)-f(x(t_{l,m};x^0,p,u),w^m,p)].$$

Here $t_{l,m}$, $l,m\in\{1,2,...,L\}$ are the points of time when the trajectory of the system (1) hits the interface of the zones X^l and X^m. The interface is defined by the equation $h_{l,m}(x)=h_{m,l}(x)=0$ with the corresponding functions $h_{l,m}(x)$ given.

Theorem 4

The components of the gradient of the target functional on the class of controls (8) are determined by the formulas:

$$\frac{\partial J(u)}{\partial L_1^m}=\frac{1}{(mesX^0\cdot mesP)}\int\limits_{X^0}\int\limits_{P}\int\limits_{\Pi_m(x^0,p,u)}\left[\frac{\partial g(x(t;x^0,p,u),u)}{\partial u}-\right.$$
$$\left.-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial u}\right]^T\cdot x^T(t;x^0,p,u)dt\cdot$$
$$\cdot\rho_P(p)\cdot\rho_{X^0}(x^0)dPdX^0,$$

$$\frac{\partial J(u)}{\partial L_2^m}=\frac{1}{(mesX^0\cdot mesP)}\int\limits_{X^0}\int\limits_{P}\int\limits_{\Pi_m(x^0,p,u)}\left[\frac{\partial g(x(t;x^0,p,u),u)}{\partial u}-\right.$$
$$\left.-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial u}\right]^T dt\cdot$$
$$\cdot\rho_p(p)\cdot\rho_{X^0}(x^0)dPdX^0,$$

where $\Pi_m(x^0,p,u)=\bigcup\limits_{m_j:x(t;x^0,p,u)\in X^m}[\tau_{m_j},\tau_{m_{j+1}}],\ j=1,2,...,$ $m=1,2,...,L$; $\psi(t;x^0,p,u)$ is the solution to the following adjoint Cauchy problem

$$\psi(\overline{T};x^0,p,u)=-\frac{\partial\Phi(x(\overline{T};x^0,p,u),\overline{T})}{\partial x},$$

$$\dot{\psi}^T(t;x^0,p,u)=\frac{\partial g(x(t;x^0,p,u),u)}{\partial x}-\psi^T(t;x^0,p,u)\cdot$$
$$\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial x}+\left[\frac{\partial g(x(t;x^0,p,u),u)}{\partial u}-\right.$$
$$\left.-\psi^T(t;x^0,p,u)\cdot\frac{\partial f(x(t;x^0,p,u),u,p)}{\partial u}\right]\cdot L_1^s,$$

for $x(t;x^0,p,u)\in X^s$ and $t\in[0,\overline{T})$, under observance of the condition (8), which satisfies the following jump condition on the interface of the zones:

$$\psi^T(t_{l,m}-0;x^0,p,u)=\psi^T(t_{l,m}+0;x^0,p,u)-$$
$$-\frac{\partial h_{l,m}(x(t_{l,m};x^0,p,u))}{\partial x}\cdot\sigma_{l,m},$$

$$\sigma_{l,m}=\frac{\psi^T(t_{l,m}+0;x^0,p,u)}{\frac{\partial h_{l,m}(x(t_{l,m};x^0,p,u))}{\partial x}\cdot f(x(t_{l,m};x^0,p,u),L^l,p)}\cdot$$
$$\cdot[f(x(t_{l,m};x^0,p,u),L^l,p)-f(x(t_{l,m};x^0,p,u),L^m,p)]$$

Here $t_{l,m}$, $l,m\in\{1,2,...,L\}$ are the points of time when the trajectory of the system (1) hits the interface of the zones X^l and X^m.

Theorem 5

For the four classes of feedback controls (5)-(8), in the event of optimizing the completion time $T=T(x^0,P)$, the derivative of the functional (3) with respect to the completion time is determined by the formula:

$$\frac{\partial J(u)}{\partial T}=\frac{\int\limits_{x^0}\int\limits_{P}\frac{\partial I(u,T;x^0,p)}{\partial T}\rho_P(p)\cdot\rho_{X^0}(x^0)dPdX^0}{(mesX^0\cdot mesP)},$$

$$\frac{\partial I(u,T;x^0,p)}{\partial T}=\frac{\partial\Phi(x(T;x^0,p,u),T)}{\partial T}+g(x(T;x^0,p,u),u(T))$$
$$-\psi^T(T;x^0,p,u)\cdot f(x(T;x^0,p,u),u(T),p),$$

where $\psi(t;x^0,p,u)$ is the solution to the adjoint problem corresponding to the case of the class of feedback controls considered.

Note that the proofs of theorems 3 and 4 use the technique of obtaining necessary first order optimality conditions as applied to discontinuous systems [7]; the proofs of theorems 1 and 2 use the technique of obtaining necessary first order optimality conditions on the classes of piecewise constant and piecewise linear functions [8]. To prove theorem 5, one can use, for example, the scheme adopted in the work [9] for problems involving a non-fixed time.

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