Stable Distributions Conforming to Kinetic Equations

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***Abstract*— It is shown that practical applicability of any noncontradictory form of probability theory is ensured by a probability model. The class of stable distributions laws conforming to kinetic equations is considered. The interpretation of stable power distributions is suggested as indicators of hierarchic or/and diffuse systems with inverse connections.**

***Keywords*— axiomatics; interpretation; probability models; stable laws; ostensive definitions; kinetic equations**

I. INTROHDUCTION

Galileo's remark of mathematics as a universal language of science became banal long ago. At the same time, the researchers who use this universal language every day suggest pairwise non-conformable statements. For example, L.D. Landau made some remarks "about truths so general that any kind of their particular application is impossible" and practically simultaneously E. Wigner published his articles about "Incomprehensible efficiency of mathematics" [1]. The authors of this report consider that contraposition of Landau's and Wigner's points of view is based on misunderstanding: mathematics is really the language of natural and other quantitative sciences. But it is just the language but not the subject, and due to the reason mentioned the contraposition of idealizations of quantitative sciences and mathematics is devoid of sense. Indeed: subject to self-consistency, the concept of validity of theory is only correct concerning idealizations accepted in the theory. The following myth is widespread: the validity of a quantitative theory axiomatized can be verified by an experiment. To reject the myth, suffice it to remind the story of Gilbert's problems. Namely: in 1900 D. Gilbert with his problem No. 6 assigned the task of axiomatization of physics and some fields of physics obtained axiomatics. The axiomatization of thermodynamics made by Carathéodory was especially elegant. At the same time, the reduction of disbalance thermodynamics and physical kinetics to the axioms of probability theory has not been crowned with success so far [4, 5]. This fall-through did not discourage the experimentalists and did not influence the rate of flow of applied publications with the use of probability theory. In the authors' judgment, the reason for the neutral attitude of practitioners to the axiomatization of probability theory is well explainable. Namely: the researchers using the probability theory /P.T./ in applied problems are convinced: the axiomatic A.N. Kolmogorov theory [2] is the ground for application of P.T. in practice, but, nevertheless, the algorithm of statistical manipulation of experimental data is based on intuitively clear ideas by Fond-Mises [3].

II. STATEMENT OF QUESTION

Kolmogorov's axiomatics eliminated a lot of ambiguities of probability theory but firstly, this axiomatics is not the only one and secondly, axiomatic theories are not at all related to practice. For instance, the statement "the value of the measurand is covered with the interval $[a,b]$ with the expectancy 0.95" has no sense within the framework of axiomatic theory. Sense (interpretation) only appears while building a probability model but the adequacy of a model is not the subject of probability theory. Mathematical language does not only give the possibility of constructing exact expressions but as well "automatizes" judgments, generates corollaries which are unexpected for the language user.

Incidentally, the nontriviality of probability theory as a dialect of mathematics is confirmed by the well-known paradox of intervals: $\{X_n\}$ is the set of normally distributed chance quantities, $S_n=X_1 + X_2 + X_3 \ldots + X_n$; in accordance with the central limit theorem

$$\lim_{n\to\infty} P(S_n < nM + x\sqrt{Dn}) = \Phi(x) .$$

Here D is variance, m is mathematical expectation, for any fixed n the following statement is true: the inequality

$$\frac{S_n - 2\sqrt{n}}{n} < m < \frac{S_n + 2\sqrt{n}}{n}$$

is performed with the probability of 95% and at the same time, the probability of performing these inequalities for all n $(\forall n)$ amounts to nothing. The paradox mentioned has the apparent explanation but the applied probability theory also has real problems [6, 7]. The authors of the report attribute to such ones the task of constructing wide enough classes of probabilistic models admitting the numerical analysis and constructive definition of the family of stable distributions. The last task is important on the reason that as a result of preconceived interpretation of "the law of large numbers" the following conviction became widespread amongst physicists and applied probability theory specialists: stationary solutions of kinetic equations asymptotically tend to distribution from class (A):

$$P(x) = \frac{\mu}{2\lambda\sigma\Gamma(\frac{1}{\mu})} \exp(-\left|\frac{x - X_c}{\lambda\sigma}\right|^{\mu}) ,$$

where

$$\lambda = \sqrt{\frac{\Gamma(\frac{1}{\mu})}{\Gamma(\frac{3}{\mu})}},$$

σ is standard deviation; X_c is the centroid of distribution coordinate, $\Gamma(z)$ is gamma-function and μ is certain constant characteristic for this distribution – its exponent. As distribution is characterized by the three parameters μ, σ and X_c, it allows to cover the wide class of well-known distributions: if $\mu < 1$ then it is Cauchy distribution; if $\mu = 1$ then it is Laplace distribution; if $\mu = 2$ then it is normal; if $\mu > 1$ then it is trapezoidal; if μ is infinite then it is proportional.

It is really a wide enough (from the point of view of practice) class of distributions but applied statistics uses other ones as well for more than 90 years. In particular, sociology, biology and some other sciences studying the epigenetics quantitatively use the distributions with densities of the kind of

$$\rho(x) = Cx^{-\alpha-1}$$

and Holtsmark and Pareto distributions with the parameter $\alpha \geq 1$

$$P\{X \geq x\} = \begin{cases} x^{-\alpha}, & x \geq 1 \\ 1, & x < 1 \end{cases}$$

and many others which are not expressed by the way of elementary functions. Characteristic property of these distributions called "Stable Laws" is

$$F(a_1 x + b_1) \otimes F(a_2 x + b_2) = F(ax + b)$$

Here $\otimes$ is the symbol of convolution product. That is, these distributions form the convolution product hemigroup. With the help of Lie group theory it is possible to construct all characteristic functions of stable laws and to give the descriptive characterization of the procedure of averaging in the terms of theory of measure. The authors do not share L.D. Landau's sarcasm but agree with the fact that the above mentioned procedure of averaging cannot be usable for a researcher. At the same time, the authors do not consider it possible to confine ourselves to "explaining just using dactylonomy" for quantitative investigation: even the distribution function is not completely evident and experimentally observed.

III. Task Solution

By the above mentioned reasons the authors define the distribution function ostensively with the help of Boltzmann type kinetic equation [5].

$$\frac{\partial f(x,t)}{\partial t} = [R]\frac{\partial \bar{k}(x) f(x,t)}{\partial x} - [R]\frac{\partial \bar{k}(x) f(x,t)}{\partial x} +$$
$$+\int\int_0^\infty\int [k(x,y;v,u) f(v,t) f(u,t) - k(x,y;v,u) f(x,t) f(y,t)] dydvdu \,. \quad (1)$$

With $[R]$ symbol we re-denoted the stream and other resource terms. Further on, we take into account that for the Maxwell-Boltzmann type systems the principle of detailed balance is implemented:

$$k(x, y; v, u) = k(v, u; x, y)\,.$$

From the symmetry of the equation kernel and detailed balance results the following:

$$\frac{\partial f(x,t)}{\partial t} + [R]\frac{\partial^2 k(x) f(x,t)}{\partial x} =$$
$$\iiint k(x,y;v,u) \times [f(v,t) f(u,t) - f(x,t) f(y,t)]\, dy\, dv\, du\,. \quad (2)$$

In the expression (2) $[R]$ only depends on time. If $[R]=R(x)$, then denoting

$$\frac{\partial [R]}{\partial x} = U$$

we get

$$\frac{\partial f(x,t)}{\partial t} + U\frac{\partial k(x) f(x,t)}{\partial x} =$$
$$\iiint k(x,y;v,u) \times [f(v,t) f(u,t) - f(x,t) f(y,t)]\, dy\, dv\, du\,. \quad (3)$$

Now we postulate obtaining the balance, then the left part is reduced to zero and we get the condition of reducing to zero for the right part: $f_{equilibrium}(v)*f_{eq.}(u)=f_{eq.}(x)*f_{eq.}(y)$

$$f_{\text{equilibrium}}(v) f_{eq}(u) = f_{eq}(x) f_{eq}(y)$$

a functional equation has the solution

$$f_{eq}(x) = ae^{-bx}.$$

So, one of equilibrium distribution functions actually belongs to class (A). Careful analysis shows that the fact

$$\frac{\partial k}{\partial x} \neq 0$$

we did not use anywhere and the equation (1) has the exponential solution only at *k*=*const*. And at $k = ax$ we get the Pareto type stationary solution. It is easy to see that the condition $k(x) = ax$ conforms the positive feedback! And that is enough for qualitative understanding of Holtsmark distribution describing the gravitating stars system. For the general case of stable power laws it is sufficient to select the kernel of the base equation (1). With this purpose in mind we use the heuristics of the work [8]

$$k(x,y;v,u) = \delta(v+u-x-y)x^n y^n e^{c(x+y-2a)}\lambda(x)\lambda(y)\lambda(v)\lambda(u); \qquad (4)$$

$$k(x,y;v,u) \neq k(v,u;x,y); \qquad (5)$$

$$k(x,t) = \alpha(t)x^{\beta} \ (\alpha \geq 0, \beta > 0); \qquad (6)$$

$$\frac{1}{\alpha(t)U}\frac{\partial f(x,t)}{\partial t} + x^{\beta}\frac{\partial f(x,t)}{\partial t} + \beta x^{\beta-1} f(x,t) = 0; \qquad (7)$$

$$\int_{\alpha}^{\infty} f(x,0)dx = 1; \qquad (8)$$

$$f(x,t)dx = \varphi(t)\psi(x), \qquad (9)$$

where

$$\varphi(0) = 1,$$

$$\frac{1}{U\alpha(t)\phi(t)}\frac{\partial\phi(t)}{\partial t} = -\left[x^{\beta}\frac{1}{\psi(x)}\frac{d\psi(x)}{dx} + \beta x^{\beta-1}\right] = const = \gamma - 1; \qquad (10)$$

$$\varphi(t) = \exp\left\{(\gamma-1)U\int_0^t \alpha(t)dt\right\}. \qquad (11)$$

Here: (5) Means the renunciation of the principle of detailed balance, moreover, (4) has the characteristic

$$k(x,y;v,u) \leq k(v,u;x,y)$$

if

$$x \leq v; u \leq y$$

in regard to condition (6) we note: we choose it from considerations of simplicity. Now let's assume that we can confine ourselves to the condition of self-consistency and using the characteristics of delta function we integrate equation 10. We get:

$$\psi(x) = \begin{cases} \frac{\gamma-1}{x^{\beta}}\left[\exp\left(\frac{\gamma-1}{(\beta-1)\alpha^{\beta-1}}\right)-1\right]^{1} \exp\left(\frac{\gamma-1}{(\beta-1)\alpha^{\beta-1}}\right), \ \beta > 1 \\ \frac{\gamma-1}{\alpha}\left(\frac{\alpha}{x}\right)^{\gamma}, \ \beta = 1 \\ \frac{\gamma-1}{x^{\beta}}\left[\exp\left(-\frac{\gamma-1}{1-\beta}(x^{1-\beta}-\alpha^{1-\beta})\right)\right], \ \beta < 1 \end{cases} \qquad (12)$$

IV. CONCLUSIONS

1) The difficulties conjugated with the notion of probability of the same type as the "number" notion which is not much added to the intuitive comprehension by Peano-Russel axiomatics.

2) Every probability model in quantitative science has the difficulty associated with the definition of the set in relation to which the averaging is performed.

3) The mechanisms resulting in exponential (a) and power (b) stable laws allow analytical description with one Boltzmann type model equation.

4) Boltzmann equation is solved by two classes of stable laws: a) exponential and b) power /including hyperbolic ones/.

5) Power distribution laws are immanent to hierarchic and/or diffuse systems with positive feedback and, consequently, can serve as their indicators.

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