The Evident Form of the Laplace Transformation of the Distribution of the First Moment Reaching the Positive Delaying Screen with the Semi-markovian Process

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***Abstract*—On the sequence of the independent, equal distributed and the positive random variables the process of the semi-markovian random walk with negative drift, positive jumps and with the positive delaying screen is constructed. In the case, when the random walk has the complex Laplace distribution, the evident form of Laplace transformation of the distribution of the first moment of reaching of the positive delaying screen with this process is found**.

Keywords—semi-markovian process of random walk with negative drift and positive jumps; Laplace transformation; duration of the drift; the size of jump.

I. INTRODUCTION

To the finding of distribution of the first moment reaching of the positive level many papers [1], [2], [3], [4] and etc. are devoted. In these papers asymptotic forms are obtained. But the evident form of the Laplace transformation of the distribution of the first moment reaching the positive delaying screen with the process semi-markovian random walk with negative drift and positive jumps in the case, if the random walk has the complex Laplace distribution in this paper is obtained.

II. SOLUTION OF THE PROBLEM

Let the sequence of the independent identically distributed and positive random variables $\{\xi_k(\omega), \zeta_k(\omega)\}_{k=\overline{1,\infty}}$. are given on the probability space $(\Omega, \Im, P(\cdot))$

We construct the following process [5]:

$$X_1(t,\omega) = z - t + \sum_{i=0}^{k-1} \zeta_i(\omega), \text{ if } \sum_{i=0}^{k-1} \xi_i(\omega) \le t < \sum_{i=0}^{k} \xi_i(\omega),$$

where $\xi_0(\omega) = \zeta_0(\omega) = 0.$

This process we delay with screen in $a \ (a > 0)$:

$$X(t,\omega) = X_1(t,\omega) - \sup_{0\le s\le t}(0, X_1(s) - a).$$

We denote

$$\tau_1^a(\omega) = \inf\{t : X(t,\omega) = a\},$$

$$L(\theta / z) = E(e^{-\theta\tau_1^a(\omega)} / X(0,\omega) = z)$$

and

$$L(\theta) = Ee^{-\theta\tau_1^a(\omega)}, \ \theta > 0.$$

Our aim to find the evident form of $L(\theta)$, when $\xi_1(\omega)$ and $\zeta_1(\omega)$ have the Erlang-n distribution of the any order and exponential distributions with parameters μ and λ accordingly.

III. THE CONSTRUCTION OF THE INTEGRAL EQUATION FOR $L(\theta / z)$

Let $X(0,\omega) = z \ge 0.$

Theorem. The Laplace transformation of the condition distribution of the random variable $\tau_1^a(\omega)$ satisfies the following integral equation

$$L(\theta / z) = \int_{s=0}^{\infty} e^{-\theta s} P\{\zeta_1(\omega) > a - z + s\} dP\{\xi_1(\omega) < s\} +$$

$$+ \int_{s=0}^{\infty} e^{-\theta s} d\ P\{\xi_1(\omega) < s\} \int_{y=z-s}^{a} L(\theta / y)\, d_y P\{\zeta_1(\omega) < y - z + s\}$$

If the random variables $\xi_1(\omega)$ and $\zeta_1(\omega)$ have the absolute-continuous distributions, then

$$L(\theta/z)=\int_{s=0}^{\infty}e^{-\theta s}P\{\zeta_1(\omega)>a-z+s\}p_{\xi_1}(s)ds+ \\ +\int_{s=0}^{\infty}e^{-\theta s}p_{\xi_1}(s)\int_{y=z-s}^{a}L(\theta/y)p_{\zeta_1}(y-z+s)dyds. \quad (1)$$

Proof. It is obvious

$$\tau_1^a(\omega)=\begin{cases}\xi_1(\omega) & ,\ if\quad z-\xi_1(\omega)+\zeta_1(\omega)>a,\\ \xi_1(\omega)+T(\omega),\ if & z-\xi_1(\omega)+\zeta_1(\omega)<a.\end{cases}$$

Then we have

$$L(\theta/z)=E(e^{-\theta\tau_1^a(\omega)}/X(0,\omega)=z)=\int_{\Omega}e^{-\theta\tau_1^a(\omega)}P_z(d\omega)=$$

$$=\int_{\{\omega:z-\xi_1(\omega)+\zeta_1(\omega)>a\}}e^{-\theta\tau_1^a(\omega)}P(d\omega)+$$

$$+\int_{\{\omega:z-\xi_1(\omega)+\zeta_1(\omega)<a\}}e^{-\theta\tau_1^a(\omega)}P(d\omega)=\int_{\{\omega:z-\xi_1(\omega)+\zeta_1(\omega)>a\}}e^{-\theta\xi_1(\omega)}P(d\omega)+$$

$$+\int_{\{\omega:z-\xi_1(\omega)+\zeta_1(\omega)<a\}}e^{-\theta[\xi_1(\omega)+T(\omega)]}P(d\omega),$$

where $T(\omega)\overset{d}{=}\tau_1^a(\omega)$.

Let $\xi_1(\omega)=s,\ \zeta_1(\omega)=y$ and $T(\omega)=x$, then we have

$$L(\theta/z)=\int_{s=0}^{\infty}e^{-\theta s}P\{\zeta_1(\omega)>a-z+s\}dP\{\xi_1(\omega)<s\}+$$

$$+\int_{s=0}^{\infty}e^{-\theta s}dP\{\xi_1(\omega)<s\}\times$$

$$\times\int_{y=z-s}^{a}L(\theta/y)d_yP\{\zeta_1(\omega)<y-z+s\}.$$

Second part of the theorem is obvious.

The theorem is proved.

IV. THE SOLUTION OF THE INTEGRAL EQUATION (1)

This equation we shall decide in the case, when the random walk has the complex Laplace distribution. The complex Laplace distribution we shall call the following distribution:

$$L_{(m^-;1^+)}(x)=P\left\{\zeta_1(\omega)-\sum_{i=1}^{m^-}\xi_i(\omega)<x\right\},\ x\in R,$$

where $\xi_i(\omega),\ i=\overline{1,m^-}$ and $\zeta_1(\omega)$ have the exponential distributions with parameters μ and λ accordingly. Then equation (1) has the following form

$$L(\theta/z)=\frac{\mu^{m-}}{(\lambda+\mu+\theta)^{m^-}}e^{-\lambda a}+ \\ +\frac{\lambda\mu^{m-}}{(m^- -1)!}e^{\lambda z}\int_{s=0}^{\infty}e^{-(\lambda+\mu+\theta)s}s^{m^- -1}\int_{y=z-s}^{a}e^{-\lambda y}L(\theta/y)\,dyds. \quad (2)$$

From equation (2) we have the following integro-differential equation

$$[L'(\theta/z)-\lambda L(\theta/z)]e^{(\mu+\theta)z}= \\ =-\frac{\lambda\mu^{m^-}}{(m^- -1)!}\int_{x=-\infty}^{z}(z-x)^{m^- -1}e^{(\mu+\theta)x}L(\theta/x)dx. \quad (3)$$

We find the m^- derivation of (3) equality

$$\sum_{i=0}^{m^-}C_{m^-}^{i}\left[L^{(i+1)}(\theta/z)-\lambda L^{(i)}(\theta/z)\right](\mu+\theta)^{m^- -i}e^{(\mu+\theta)z}= \\ =-\lambda\mu^{m^-}e^{(\mu+\theta)z}L(\theta/z). \quad (4)$$

From (4) we have differential equation

$$\sum_{i=0}^{m^-}C_{m^-}^{i}\left[L^{(i+1)}(\theta/z)-\lambda L^{(i)}(\theta/z)\right](\mu+\theta)^{m^- -i}+ \\ +\lambda\mu^{m^-}L(\theta/z)=0$$

with characteristic equation

$$\sum_{i=0}^{m^-}C_{m^-}^{i}\left[k^{i+1}(\theta)-\lambda k^{i}(\theta)\right](\mu+\theta)^{m^- -i}+\lambda\mu^{m^-}=0, \quad (5)$$

with common solution

$$L(\theta/z)=\sum_{i=1}^{m^-}c_i(\theta)e^{k_i(\theta)z}, \quad (6)$$

where $k_i(\theta),\ i=\overline{1,m^-}$ are the roots of the characteristic equation (5), with boundary conditions

$$
\begin{cases}
L(\theta/0) = \dfrac{\mu^{m^-} e^{-\lambda a}}{(\lambda+\mu+\theta)^{m^-}} + \\
\dfrac{\lambda\mu^{m^-}}{(m^- - 1)!}\displaystyle\int_{s=0}^{\infty} s^{m^- -1} e^{-(\lambda+\mu+\theta)s} \int_{x=-s}^{a} e^{-\lambda x} L(\theta/\boldsymbol{x})\,dx\,ds, \\
L'(\theta/0) = \lambda L(\theta/0) + \\
+(-1)^{m^-} \dfrac{\lambda\mu^{m^-}}{(m^- -1)!}\displaystyle\int_{x=-\infty}^{0} x^{m^- -1} e^{(\mu+\theta)x} L(\theta/x)\,dx, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\displaystyle\sum_{i=0}^{k} C_k^i \left[L^{(i+1)}(\theta/0) - \lambda L^{(i)}(\theta/0)\right](\mu+\theta)^{k-i} = \\
= \dfrac{(-1)^{m^- -k}\lambda\mu^{m^-}}{(m^- -(k+1))!}\displaystyle\int_{x=-\infty}^{0} x^{m^- -(k+1)} e^{(\mu+\theta)x} L(\theta/x)\,dx, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\displaystyle\sum_{i=0}^{m^- -1} C_{m^- -1}^i \left[L^{(i+1)}(\theta/0) - \lambda L^{(i)}(\theta/0)\right](\mu+\theta)^{m^- -1-i} = \\
= -\lambda\mu^{m^-}\displaystyle\int_{x=-\infty}^{0} e^{(\mu+\theta)x} L(\theta/x)\,dx.
\end{cases}
$$

If take into consideration the common solution, we have the system of the linear algebraic equations. After some complex transformation this system reduced to one equation. Only in the solution

$$
\left(c_1(\theta), c_2(\theta), \ldots, c_{m^-+1}(\theta)\right) = \left(\frac{\mu^{m^-}}{\left[\mu+\theta+k_1(\theta)\right]^{m^-}} e^{-k_1(\theta)a}, 0, \ldots, 0\right)
$$

.

(6) will be the Laplace transformation of the distribution of the random variable $\tau_1^a(\omega)$.

Then we have

$$
L(\theta/z) = \frac{\mu^{m^-}}{\left[\mu+\theta+k_1(\theta)\right]^{m^-}} e^{-k_1(\theta)(a-z)}.
$$

According to form of the total probability for expected value we have

$$
\begin{aligned}
L(\theta) &= \int_{x=0}^{\infty} L(\theta/a-x)\,dP\{X(0,\omega) < x\} = \\
&= \frac{\mu^{2m^-}}{\left[\mu-k_1(\theta)\right]^{m^-}\left[\mu+\theta+k_1(\theta)\right]^{m^-}} e^{-k_1(\theta)a}
\end{aligned} \tag{7}
$$

From (7) we have

$$
\begin{aligned}
E\tau_1^a(\omega) &= -\frac{m^-\lambda}{m^-\lambda-\mu}a + \frac{m^-}{\mu} = \\
&= -\frac{E\xi_1(\omega)}{E\xi_1(\omega) - E\zeta_1(\omega)}a + E\xi_1(\omega),
\end{aligned}
$$

$$
\begin{aligned}
D\tau_1^a(\omega) &= -\frac{m^-(m^-+1)\lambda\mu}{(m^-\lambda-\mu)^3}a + \frac{m^-}{(m^-\lambda-\mu)^2} = \\
&= -\frac{(m^-+1)\left[E\xi_1(\omega)\right]^2\left[E\zeta_1(\omega)\right]^2}{m^-\left[E\xi_1(\omega)-E\zeta_1(\omega)\right]^3}a + \frac{[E\xi_1(\omega)]^2[E\zeta_1(\omega)]^2}{m^-\left[E\xi_1(\omega)-E\zeta_1(\omega)\right]^2}.
\end{aligned}
$$

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