One Model of Risky Asset Price Evolution Described by Gaussian Martingale

Omar Glonti[1], Zaza Khechinashvili[2]
I.Javakhishvili Tbilisi State University, Tbilisi, Georgia
[1]omglo@yahoo.com, [2]khechinashvili@gmail.com

Abstract - We propose a model of stock price evolution with Gaussian martingale, investigate it's properties and consider the problem of optimal in mean square sense forecasting for this model.

Keywords - Risky asset modeling, Gaussian martingale, forecasting,

On the probability space $(\Omega, F, (F_n)_{0 \le n \le N}, P)$ we consider the following real valued stochastic process in discrete time as a model of stock price evolution

$$S_n = S_0 e^{H_n}, \qquad (1)$$

where S_0 is non random and positive and

$$H_n = \sum_{n=1}^{n} h_n \qquad (2)$$

with

$$h_n = \sigma_n \Delta M_n, \qquad (3)$$

$$\sigma_n = a_n + e^{-b_n M_{n-1}}. \qquad (4)$$

Here $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ are positive sequences of real numbers, $M = (M_n, F_n), n = 0,1,2,..., N,$ is the Gaussian martingale with quadratic characteristic $\langle M \rangle_n = EM_n^2$.

In this paper we investigate some properties of proposed model (1)-(4). The following results are obtained:

Theorem 1. In scheme (1)-(4) the covariance $\operatorname{cov}(h_{n-1}, \sigma_n)$ between logarithmic return h_{n-1} at moment $n-1$ and volatility σ_n at next moment n is negative for any $n = 0,1,2,..., N$, i.e. model (1)-(4) has fixed so called leverage effect.

Proof . It is clear, that from (1)-(4)

$$\begin{aligned} Eh_n &= E[\sigma_n \Delta M_n] = E[(a_n + e^{-b_n M_{n-1}}) \times \\ \Delta M_n] &= E[E(a_n + e^{-b_n M_{n-1}}) \Delta M_n / F_{n-1}] = \\ &E[(a_n + e^{-b_n M_{n-1}}) E[\Delta M_n / F_{n-1}]] = 0 \end{aligned} \qquad (5)$$

for any n and

$$\begin{aligned} &\operatorname{cov}(h_{n-1}, \sigma_n) = Eh_{n-1}\sigma_n - \\ &Eh_{n-1}E\sigma_n = Eh_{n-1}\sigma_n \end{aligned}. \qquad (6)$$

Using properties of Gaussian martingale we obtain

$$\begin{aligned} &Eh_{n-1}\sigma_n = E[(a_{n-1} + e^{-b_{n-1}M_{n-2}})\Delta M_{n-1} \times \\ &(a_n + e^{-b_n M_{n-1}})] = a_{n-1}a_n E[\Delta M_{n-1}] + \\ &a_n E[e^{-b_{n-1}M_{n-2}} \Delta M_{n-1}] + a_{n-1} \times \\ &E[e^{-b_n M_{n-1}} \Delta M_{n-1}] + \\ &E[e^{-(b_{n-1}M_{n-2} + b_n M_{n-1})} \Delta M_{n-1}] = \\ &a_n E[e^{-b_{n-1}M_{n-2}} E[\Delta M_{n-1} / F_{n-2}]] + \\ &a_{n-1} E[e^{-b_n M_{n-2}} E[e^{-b_n \Delta M_{n-1}} \times \\ &\Delta M_{n-1} / F_{n-2}]] + E[e^{-(b_{n-1}+b_n)M_{n-2}} \times \\ &E[e^{-b_n \Delta M_{n-1}} \Delta M_{n-1} / F_{n-2}]] = \\ &a_{n-1} E[e^{-b_n \Delta M_{n-2}}] E[e^{-b_n \Delta M_{n-1}} \Delta M_{n-1}] + \\ &E[e^{-(b_{n-1}+b_n)M_{n-2}}] \times E[e^{-b_n \Delta M_{n-1}} \Delta M_{n-1}] = \\ &[a_{n-1} E[e^{-b_n \Delta M_{n-2}}] + E[e^{-(b_{n-1}+b_n)M_{n-2}}]] \times \\ &E[e^{-b_n \Delta M_{n-1}} \Delta M_{n-1}]. \end{aligned} \qquad (7)$$

Note, that ΔM_{n-1} has normal distribution with mean 0 and variance $\Delta \langle M \rangle_{n-1}$ for each n and

$$\begin{aligned} &E[e^{-b_n \Delta M_{n-1}} \Delta M_{n-1}] = \frac{1}{\sqrt{2\pi \Delta \langle M \rangle_{n-1}}} \times \\ &\int_{-\infty}^{\infty} e^{-b_n x} x e^{-\frac{x^2}{2\Delta \langle M \rangle_{n-1}}} dx = \\ &e^{\frac{b_n^2 \Delta \langle M \rangle_{n-1}}{2}} \frac{1}{\sqrt{2\pi \Delta \langle M \rangle_{n-1}}} \times \\ &\int_{-\infty}^{\infty} x e^{-\frac{(x + b_n \Delta \langle M \rangle_{n-1})^2}{2\Delta \langle M \rangle_{n-1}}} dx = \\ &e^{\frac{b_n^2 \Delta \langle M \rangle_{n-1}}{2}} (-b_n \Delta \langle M \rangle_{n-1}) < 0 \end{aligned} \qquad (8)$$

because $b_n > 0$ for each n.

From (5),(6) and (7)it is clear that $\mathrm{cov}(h_{n-1}, \sigma_n) < 0$.

The proof of Theorem 1 is completed.

Note, that this effect was discovered by F.Black in 1976 for real financial time series, and the sense of this effect is that falling down of return implies increasing of volatility (see [1] about EGARCH,TGARCH, HARCH-models).

Theorem 2. In scheme (1)-(4) for any moment of time n kurtosis coefficient k_n of logarithmic return h_n is positive.

Proof . We know (see (5)) that $Eh_n = 0$ for each n and therefore kurtosis coefficient

$$k_n = \frac{Eh_n^4}{(Eh_n^2)^2} - 3. \tag{9}$$

At first we find Eh_n^2:

$$Eh_n^2 = E[\sigma_n^2(\Delta M_n)^2] = E[\sigma_n^2 E(\Delta M_n)^2 / F_{n-1}] = E\sigma_n^2 E(\Delta M_n)^2. \tag{10}$$

Then

$$Eh_n^4 = E[\sigma_n^4(\Delta M_n)^4] = E[\sigma_n^4 E(\Delta M_n)^4 / F_{n-1}] = E\sigma_n^4 E(\Delta M_n)^4 \tag{11}$$

and from (9) using (10) and (11) we obtain

$$k_n = \frac{E\sigma_n^4}{(E\sigma_n^2)^2}\frac{E(\Delta M_n)^4}{[E(\Delta M_n)^2]^2} - 3, \tag{12}$$

but ΔM_n has the normal distribution with expectation 0 and kurtosis coefficient 0, so

$$\frac{E(\Delta M_n)^4}{[E(\Delta M_n)^2]^2} = 3. \tag{13}$$

Using (13) from (12) we obtain, that

$$k_n = 3\frac{E\sigma_n^4}{(E\sigma_n^2)^2} - 3 = 3\left(\frac{E\sigma_n^4}{(E\sigma_n^2)^2} - 1\right) = 3\frac{E\sigma_n^4 - (E\sigma_n^2)^2}{(E\sigma_n^2)^2} > 0$$

and the proof of Theorem 2 is completed.

It is known, that (see [1]-[3]) for real financial time series the empirical kurtosis coefficient of logarithmic return is usually positive.

Forecasting. Consider the forecasting of stock price $S = (S_n, F_n),\ n = 0,1,\ldots, N,$ described by (1)-(4) on the one step, i.e. the problem of finding $\hat{S}_n(1) = E(S_n / F_{n-1}^S),$ where $F_n^S = \sigma\{S_k, k \le n\}$.

From (1)-(4) we have

$$S_n = S_{n-1}\exp\{(a_n + e^{-b_n M_{n-1}})\Delta M_n\}.$$

Then

$$\begin{aligned}
\hat{S}_n(1) &= E(S_n / F_{n-1}^S) = S_{n-1}E[\exp\{(a_n + e^{-b_n M_{n-1}})\times \\
&\Delta M_n\} / F_{n-1}^S] = S_{n-1}E[E[\exp\{(a_n + e^{-b_n M_{n-1}})\times \\
&\Delta M_n\} / F_{n-1}] / F_{n-1}^S] = S_{n-1}E[E[\exp\{(a_n + e^{-b_n x})\times \\
&\Delta M_n\}]_{x=M_{n-1}} / F_{n-1}^S] = S_{n-1}E[\exp\{(a_n + e^{-b_n x})\times \\
&\Delta M_n\}]_{x=M_{n-1}} = S_{n-1}\exp\{(a_n + e^{-b_n M_{n-1}})^2\frac{\Delta\langle M\rangle_n}{2}\},
\end{aligned}$$

where $\langle M\rangle_n = EM_n^2$ and it is clear, that $M_n = M_n(S)$ is F_n^S -measurable because

$$M_n = M_{n-1} + \ln\frac{S_n}{S_{n-1}}(a_n + e^{-b_n M_{n-1}(S)})^{-1}, M_0 = 0.$$

Therefore

$$\hat{S}_n(1) = S_{n-1}\exp\{(a_n + e^{-b_n M_{n-1}})^2\frac{\Delta\langle M\rangle_n}{2}\}, S_0 > 0.$$

Remark 1. It is not difficult to obtain the forecasting of S on m step, i.e. $\hat{S}_n(m) = E(S_n / F_{n-m}^S)$, $m < n$ using the representation following from (1)-(4)

$$S_n = S_m \exp\{\sum_{k=m+1}^{n}(a_n + e^{-b_n M_{n-1}})\Delta M_n\}$$

Remark 2. In our financial market described by (1)-(4) consider risky asset with the following price evolution

$$P_n = \exp\{\sum_{k=1}^{n}(a_k - e^{-b_k M_{k-1}})^2\frac{\Delta\langle M\rangle_k}{2}\}, n = 0,1,\ldots, N.$$

If we choose P_n as an numeracies we obtain that $\left(\frac{S_n}{P_n}, F_n\right)$ is a martingale.

References

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