Consistent Criteria in Metric Space

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Abstract— **In this paper, we propose the consistent criteria of hypothesis verification with zero probability of errors of any kind. we prove necessary and sufficient conditions for the existence such consistent criteria in metric space.**

Keywords— consistent criteria; metric space; measure

Let (E,S) be a measurable sample space with a given family of probability measures $\{\mu_h, h \in H\}$ where H is a set of hypotheses. Let $\mathrm{B}(H)$ is a σ - algebra which contains all finite subsets of H .

Definition 1. The family of probability measures $\{\mu_h, h \in H\}$ will be said to admit a consistent criterion of hypotheses if there exist even though one measurable map δ of the space (E,S) in $(H, B(H))$ such that $\mu_h(x : \delta(x) = h) = 1, \forall h \in H$.

Note 1. If $y \in \{x : \delta(x) = h_k\}$, then we receive h_k hypothesis.

Definition 2. For given δ criterion we say that the mistake is of the i kind, if H_i hypothesis is repudiated when it is true.

Definition 3. The following probability

$$\alpha_i(\delta) = \mu_{h_i}(x : \delta(x) \neq h_i)$$

is called the probability of mistake of the i kind δ criterion.

Note 2. If the family of probability measures $\{\mu_h, h \in H)$ admits a consistent criterion of hypotheses, so the probability of mistakes of all kind will be zero.

Let E and H be a complete metric space, $\delta : (E,S) \to (H, B(H))$ is continuous consistent criterion. If $F_1, F_2 \subset H$ are two closed disjoint sets and ν_1, ν_2 measures focused (gathered) on F_1, F_2 , respectively, for the measures:

$$\mu_{F_i}(A) = \int_{F_i} \mu_h(A)\nu_i(dh), \quad i = 1,2 \qquad (1)$$

then there are two disjoint closed sets $E_{F_1}, E_{F_2} \subset E$, which on these measures are focused (gathered). Suppose the contrary, the measures (1) what were disjoint closed sets F_1, F_2 and measures $\nu_i(F_j) = 0, \ i \neq j$, have disjoint closed carriers:

$$E_{F_i} \subset E, i = 1,2; \mu_{F_i}(F_j) = 0$$

at $i \neq j$.

Choosing the ν_i measure so that $\nu_i(F_i \cap U) > 0$, if $U \subset H$ an open set and $F_i \cap U$ is not empty, then we assume that $\mu_k(E_F) = 1$ is true for almost all h , but then by the continuity μ_k from above. Let $F \subset H$ a closed set, and $F_n, n = 1,2,...$ such increasing sequence of sets that $F = H \setminus \bigcup_n F_n$. For each n , we can specify a pair $E'_n, E''_n \subset E$, of closed sets, such that:

$$\mu_h(E'_n) = 1, h \in F; \ \ \mu_h(E''_n) = 1, h \notin F_n; \ \ E'_n \cap E''_n = \Theta$$

Let $E_F = \bigcap_n E_n$. It is clear that F is a closed set and

$$\mu_h(E_F) = 1, \forall h \in F, \mu_h(E_F) = 0, h \notin F .$$

So, for each closed $F \subset H$ there exists a closed set E_F , so that

$$\mu_h(E_F) = \chi_F(h), \qquad (2)$$

where $\chi_F(h)$ the indicator of set F . Use this result, we investigate the existence of a continuous consistent criteria.

Theorem: In order for a family $\{\mu_h, h \in H\}$ of measures on complete separable metric space (E,S) , where H is a complete metric space, necessary and sufficient condition for exist once of continuous consistent criterion δ is, that measures (1) for any disjoint closed sets $F_1, F_2 \subset H$ have disjoint closed supports.

Proof: The necessity of the theorem established above. In order to prove of sufficient of the condition use (2). Denote also by E_F the minimal closed set for which have place (2). It is obtained from E_F by throw out all balls $B \subset E$ for whom $\mu_k(B) = 0, \forall h \in F$. If we assume, that each set F corresponds set E_F, then $F_1 \subset F_2$ and $E_{F_1} \subset E_{F_2}$ (so far as). $E_{F_1} \subset E_{F_2} \cup E_{F_2}$ will be $E_{F_1 \cup F_2} = E_{F_1} \cup E_{F_2}$

If $F_n \uparrow F$, then E_F coincides with the closure of $\bigcup_n E_{F_n}$. Obviously, that $F_1 \cap F_2 = \Theta$ and $E_{F_1} \cap E_{F_2} = \Theta$. Denote with $U_k^{(n)}, K = 1,2,...$ family of closed bolls of radius 2^{-n} , which centers formed 2^{-n-1} net in H . Than $H = \cup U_k^{(n)}$ and every point h is inner although bi to one of the sets $U_k^{(n)}$.

Suppose also, that for every $U_k^{(n+1)}$ there is $U_k^{(n)} \subset U_k^{(n+1)}$. We denote by $E_F^{(n)}$ a closed set in E , corresponding to set $U_k^{(n)}$.

Let E_h be the corresponding to one-point set $\{h\}$ It is easy, to see that for each closed set F have place a correspondence: E_F is a closure for $\bigcup_{h \in F} E_h$. Let $\delta(x)$ is defined by equality $\delta(x) = h$, when $x \in E_h$. The map δ defined on a dense set in E_H . We extend it to $E_H \setminus \bigcup_{h \in F} E_h$.

If, for some n there is k , that if $x \in E_k^{(n)}$, then $x \in \bigcup_n E_{k_n}^{(n)}$, where k_n a sequence of natural numbers. Let us assume $\delta(x) = h$, where $h = \bigcap_n U_{k_n}^{(n)}$ .This intersection is not empty because $\bigcap_n E_{k_n}^{(n)}$ is non-empty and consists for a single point by the completeness of the space.

Suppose, that for some m ,

$$x_0 \in E_{k_m}^{(n)}, \ x_0 \notin \bigcup_k E_k^{(m+1)} .$$

If $x_i \in E_{h_i}$ and $x_i \to x_0$ then h_i it has no limit points. Indeed, if $h_i \to h_0$ and h_0 - an interior point in $U_{k_m+1}^{(m)}$, then $E_{h_i} \subset U_{k_m+1}^{(m+1)}$, for sufficiently large i $E_{h_i} \subset E_{k_m+1}^{(m+1)}, x_i \in E_{k_m+1}^{(m+1)}, x_0 = \lim x_i \in E_{k_{m+1}}^{(m+1)}$.

Let $h_k^{(m)} \in U_k^{(m)}$ an arbitrary point. Assuming $\delta(x_0) = h_k^{(m)}$, we define $\delta(x)$ for all $x \in \bigcup_k E_k^{(1)}$. Finally, $\delta(x) = h_0$ for $x \in E_h \setminus \bigcup_k E_k^{(1)}$, where h_0 - an arbitrary fixed value. From the construction $\delta(x)$ it follows that the prototype of $U_k^{(n)}$ for all k and n is $E_k^{(n)}$. Therefore, for every open set V we have $\delta^{-1}(V) = \bigcup_{U_k^{(n)} \subset V} E_k^{(n)}$, consequently $\delta(x)$ is measurable. Further $\delta^{-1}(H \setminus V) = \bigcap_{U_k^{(n)} \subset V} (E_H \setminus E_k^{(n)})$.

If $H \setminus V = F$, then $E_k^{(n)} \cap E_F = \Theta$ when $U_k^{(n)} \subset V$. This means $\delta^{-1}(F) \supset E_F$.

Since $\delta^{-1}(F)$ and E_F contain a dense subset $\bigcup_{h \in F} E_h$, then E_F is dense in $\delta^{-1}(F)$ and hence $E_F = \delta^{-1}(F)$.

The theorem is proved.

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