Numerical Solution One Nonlocal Problem

Galina Mehdiyeva[1], Aydin Aliyev[2]
Baku State University, Baku, Azerbaijan
[1]*imn_bsu@mail.ru*, [2]*aydin_aliyev66@mail.ru*

Abstract— In the rectangular area is considered a nonlocal boundary value problem for the Laplace equation. For solutions used discrete analog of the Fourier method. Effectively estimated error of the method.

Keywords— *nonlocal problem; difference scheme; rectangle; differentiable function*

I. Formulation of the Problem

Let us denote through Π a rectangle with vertices (0,0), (1,0), (1,b), (0,b), where b-rational number. Let Γ-boundary of this rectangle.

We introduce net square by lines

$$x = x_i = ih, \quad y = y_j = jh \ \ (i = 0,1,...,1/h, j = 0,1,...,b/h),$$

where $1/h$ and b/h are integer numbers. Let

$$\Pi_h = \begin{Bmatrix} (x,y): x = x_i = ih, i = 0,1,...,1/h, \\ y = y_j = jh, j = 0,1,...,b/h \end{Bmatrix},$$

and Γ_h - set of net knots, lying on Γ.

Consider the following nonlocal problem

$$\left.\begin{array}{l} \Delta u = 0 \qquad in \ \ \Pi, \\ u(x,0) = u(x,b) = 0 \qquad (0 < x < 1), \\ u(1,y) = 0, \\ u(0,y) = \alpha u(c,y) + f(y) \ \ (0 < y < b), \end{array}\right\} \tag{1}$$

where $f(y)$ are five times continuously differentiable function and

$$f(0) = f(b) = 0.$$

II. The Difference Problem

We build corresponding difference scheme in following way:

$$\left.\begin{array}{l} \Delta_h u_h = 0 \qquad in \ \ \Pi_h, \\ u_h(x,0) = u_h(x,b) = 0 \qquad (0 < x < 1), \\ u_h(1,y) = 0, \\ u_h(0,y) = \alpha u_h(c,y) + f(y) \ \ (0 < y < b). \end{array}\right\} \tag{2}$$

Suppose that $x = c$ coincides with on x_i points.

It can be easily verified that the solutions of problems (1) and (2) are defined accordingly by formulas

$$u(x,y) = \sum_{n=1}^{\infty} c_n g(x, n\pi) \sin\frac{n\pi y}{b},$$

$$u_h(x,y) = \sum_{n=1}^{1/h} \gamma_n g(x, \beta_n / h) \sin\frac{n\pi y}{b},$$

where

$$c_n = \frac{2}{b}\int_0^b f(t)\sin\frac{n\pi t}{b}dt,$$

$$\gamma_n = \frac{2h}{b}\sum_{r=1}^{1/h} f(rh)\sin\frac{n\pi rh}{b},$$

$$g(x,z) = \frac{sh(1-x)\frac{z}{b}}{sh\frac{z}{b} - \alpha\, sh(1-c)\frac{z}{b}}$$

and β_n is defined from

$$sh\frac{\beta_n}{2b} = \frac{\sin nh\pi/2b}{\sqrt{1-\frac{2}{3}\sin^2 nh\pi/2b}} \tag{3}$$

III. Construct a Difference Scheme

Easy to see that $u_h(x,y)$ satisfies boundary conditions. We prove that it satisfies the difference scheme. Hence

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$$u_h(x-y,y\pm h)=\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\times$$
$$\times\sin\frac{n\pi y}{b}\cos\frac{n\pi h}{b}\pm$$
$$\pm\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\sin\frac{n\pi h}{b}\cos\frac{n\pi y}{b}\pm$$
$$\pm\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\cos\frac{n\pi y}{b}\sin\frac{n\pi h}{b}\pm$$
$$\pm\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\cos\frac{n\pi y}{b}\sin\frac{n\pi h}{b},$$

$$u_h(x+h,y+h)+u_h(x-h,y-h)=$$
$$=2\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\cos\frac{n\pi h}{b}\sin\frac{n\pi y}{b}+$$
$$+2\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\sin\frac{n\pi h}{b}\cos\frac{n\pi y}{b},$$

$$u_h(x+h,y)+u_h(x,y+h)+u_h(x-h,y)+$$
$$+u_h(x,y-h)-4u_h(x,y)=2\sum_{n=1}^{1/h}\gamma_n g(x,\frac{\beta_n}{h})\times$$
$$\times[\cos\frac{n\pi h}{b}+ch\frac{\beta_n}{b}-2]\sin\frac{n\pi y}{b},$$

$$u_h(x-y,y+h)=\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\times$$
$$\times\cos\frac{n\pi h}{b}\sin\frac{n\pi y}{b}+$$
$$+\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\sin\frac{n\pi h}{b}\cos\frac{n\pi y}{b}-$$
$$-\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\cos\frac{n\pi h}{b}\sin\frac{n\pi y}{b}-$$
$$-\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\sin\frac{n\pi h}{b}\cos\frac{n\pi y}{b},$$

$$u_h(x+y,y-h)=\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\times$$
$$\times\cos\frac{n\pi h}{b}\sin\frac{n\pi y}{b}-$$
$$-\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\sin\frac{n\pi h}{b}\cos\frac{n\pi y}{b}+$$
$$+\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\cos\frac{n\pi h}{b}\sin\frac{n\pi y}{b}-$$
$$-\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\sin\frac{n\pi h}{b}\cos\frac{n\pi y}{b},$$

$$u_h(x+h,y-h)+u_h(x-h,y+h)=$$
$$=2\sum_{n=1}^{1/h}\gamma_n\frac{sh(1-x)\frac{\beta_n/h}{b}ch\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\cos\frac{n\pi h}{b}\sin\frac{n\pi y}{b}-$$
$$-2\sum_{n=1}^{1/h}\gamma_n\frac{ch(1-x)\frac{\beta_n/h}{b}sh\frac{\beta_n}{b}}{sh\frac{\beta_n/h}{b}-\alpha sh(1-c)\frac{\beta_n/h}{b}}\sin\frac{n\pi h}{b}\cos\frac{n\pi y}{b},$$

$$\Delta_h u_h=4\sum_{n=1}^{1/h}\gamma_n g(x,\frac{\beta_n}{h})[ch\frac{\beta_n}{b}\cos\frac{n\pi h}{b}+$$
$$+2\cos\frac{n\pi h}{b}+2ch\frac{\beta_n}{b}-5]\sin\frac{n\pi y}{b}.$$

Prove that

$$(ch\frac{\beta_n}{b}+2)\cos\frac{n\pi h}{b}+2ch\frac{\beta_n}{b}-5=0 \tag{4}$$

Hence

$$(ch\frac{\beta_n}{b}+2)\cos\frac{n\pi h}{b}+2ch\frac{\beta_n}{b}-5=3ch\frac{\beta_n}{b}-3-$$
$$-2\sin^2\frac{n\pi h}{2b}(ch\frac{\beta_n}{b}+2)=3sh^2\frac{\beta_n}{2b}-\sin^2\frac{n\pi h}{2b}\times$$
$$\times(2sh^2\frac{\beta_n}{2b}+3)=3sh^2\frac{\beta_n}{2b}-2\sin^2\frac{n\pi h}{2b}sh^2\frac{\beta_n}{2b}-$$
$$-3\sin^2\frac{n\pi h}{2b}=[3-2\sin^2\frac{n\pi h}{2b}]sh^2\frac{\beta_n}{2b}-3\sin^2\frac{n\pi h}{2b}.$$

Considering

$$sh\frac{\beta_n}{2b}=\frac{\sin\frac{n\pi h}{2b}}{\sqrt{1-\frac{2}{3}\sin^2\frac{n\pi h}{2b}}}$$

we get the validity of (4).

IV. Estimate of the Error

Hence

$$|c_n|\le kn^{-5}, \tag{5}$$

where

$$k=\frac{2b^4}{\pi^5}[|f^{(IV)}(b)|+|f^{(IV)}(0)|]+\frac{4b^5}{\pi^6}\max|f^{(V)}(t)|.$$

It can be easily proved that

$$|\beta_n-nh\pi|\le\frac{(nh\pi)^5}{480b^4}. \tag{6}$$

We have

$$\left|\frac{\partial g(x,z)}{\partial z}\right|\le\frac{1}{16b}(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4}{3b}))^{-2}\times$$
$$\times[x\exp(-\frac{x}{b}z)+\alpha(x+c)\exp(-\frac{x+c}{b}z)],$$

in $1\le n\le 1/h, 0\le y\le b, \sqrt{3}n\pi\ge z\ge\frac{\beta_n}{h}$.

Then, using (6), we have

$$|g(x,\beta_n/h)-g(x,n\pi)|\le\frac{1}{16b}(1-\exp(-\frac{8}{3b})-$$
$$-\alpha\exp(-\frac{4}{3b}))^{-2}[x\exp(-\frac{x}{b}z)+$$
$$+\alpha(x+c)\exp(-\frac{(x+c)}{b}z)]\frac{(n\pi)^5}{480b^4}h^4\le \tag{7}$$
$$\le\frac{\pi^5}{7680b^5}(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4}{3b}))^{-2}\times$$
$$\times[x\exp(-\frac{4x}{3b}n)+\alpha(x+c)\exp(-\frac{4(x+c)}{3b}n)]n^5h^4$$

At last not that

$$0\le g(x,z)\le\frac{1}{1-\alpha}\quad(0\le x\le 1,\ z\ge\frac{4b}{3}). \tag{8}$$

Now estimate $|u-u_h|$. We have

$$|u-u_h|\le R_1+R_2,$$

where

$$R_2=\sum_{n=1+1/h}^{\infty}|c_n|g(x,n\pi).$$

It follows from (5) and (8) that

$$R_1=\sum_{n=1}^{1/h}|c_n||g(x,\beta_n/h)-g(x,n\pi)|,\quad R_2\le K\sum_{n=1+1/h}^{\infty}n^{-5}\le K\frac{h^4}{2}.$$

Using (5) and (7), we received

$$|R_1|\le K\sum_{n=1}^{1/h}n^{-5}[x\exp(-\frac{4nx}{3b})+\alpha(x+c)\exp(-\frac{4n(x+c)}{3b})]\times$$
$$\times(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4c}{3b}))^{-2}\frac{\pi^5}{7680b^5}n^5h^4=$$
$$=K\frac{\pi^5}{7680b^5}h^4(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4c}{3b}))^{-2}\times$$

$$\times\sum_{n=1}^{1/h}[x\exp(-\frac{4nx}{3b})+\alpha(x+c)\exp(-\frac{4n(x+c)}{3b})]\le$$

$$\le\frac{K\pi^5h^4}{10240b^4}(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4c}{3b}))^{-2}(1+\alpha),$$

$$|u-u_h|\le K\frac{h^4}{2}+\frac{K\pi^5}{10240b^4}(1+\alpha)(1-\exp(-\frac{8}{3b})-$$
$$-\alpha\exp(-\frac{4c}{3b}))^{-2}h^4,$$

$$|u-u_h|\le K[0{,}5+\frac{\pi^5}{10240b^4}(1+\alpha)\times$$
$$\times(1-\exp(-\frac{8}{3b})-\alpha\exp(-\frac{4c}{3b}))^{-2}]h^4$$

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