Investigation of Free Vibrations of Viscoelastic Bodies

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***Abstract*— The method of the solution of problems of free fluctuations of viscoelastic elements of designs from a linear viscoelastic material Is developed at any hereditary kernels. Expressions of frequency and factor of attenuation of viscoelastic fluctuations which an averaging method were approximately received for the first time by the outstanding scientist of the XX century A.A.Ilyyushin and his employees are specified. The method is based to an original approach of calculation of poles of subintegral function in Mellin's formula at any, not set analytically, hereditary kernel of a relaxation. It open a way of application of integrated transformation of Laplas to the solution of non-stationary dynamic problems of a viscoelastic with real rheological properties.**

***Keywords*— viscoelasticy; structural elements; free vibrations; hereditary kernels; frequency; damping facto**

I. INTRODUCTION

The recently observed intensive introduction of new materials – polymer –based, nanocomposites, in contemporary machine and instrument engineering caused a great interest for studying the dependence of their physical -mechanical properties on the internal structure. As is known, the synthesis of materials with the given physical - mechanical properties concerns the rank of "eternal" problems of mechanics of materials and material science. These problems became especially urgent in the last two decades when the structure of the material could be controlled on the level of separate molecules and even atoms. Deviation of quality characteristic of structural components is a very important problem without which quality description and prediction of the properties of polymer nanocomposites may not be performed. Similar problems were solved in the papers of the authors [2-8] in different statements and by other methods. The solution of one-dimensional dynamical problems of linear viscoelasticity under arbitrary hereditary kernels vere first investigated in detail in the papers of M. Kh. Ilyasov [9].

II. PROBLEM STATEMENT

For investigating free vibrations of viscoelastic bodies we'll solve the homogeneous equation

$$T' + \omega^2 T = \varepsilon\omega^2 \int_0^1 \Gamma(t-\tau)T(\tau)d\tau, \quad t>0 \qquad (1)$$

with the following initial conditions

$$T(0) = T_0, \;\; T(0) = T_1 \qquad (2)$$

Using the Laplace transformation, we get the following representation of the solution of problem (1), (2)

$$\bar{T}\,(p) = \frac{pT_0 + T_1}{p^2+\omega^2-\varepsilon\omega^2\bar{\Gamma}\,(p)} \qquad (3)$$

where the dash over the letter indicates integral transformations of the functions having the same name, for instance, $\bar{T}\,(p)$ means the Laplace integral

$$\bar{T}\,(p) = \int_0^\infty T(t)\,e^{-pt}dt$$

p is a complex parameter of transformation. The function $\bar{T}\,(p)$ represented by formula (3), and the transform of the relaxation kernel $\bar{\Gamma}\,(p)$ are analytic in the Wight half-plan *Re p>0* of the complex plane *p*. It we denote $x_1(t) = T(t)$ and $x_2(t) = T'(t)$, we can write problem (1)-(2) in the vector form

$$x'(t) = Yx(t) + \varepsilon\omega^2 Y_1 \int_0^1 \Gamma(t-s)x(s)ds, \quad x(0) = x_0 \qquad (4)$$

where $x(t) = col\big(x_1(t),\; x_2(t)\big)$,

$$Y = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \;\; Y_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad x_0 = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix}.$$

Applying to equation (4) then Laplace transform, we get

$$p\bar{x} - x_0 = \mathrm{Y}\bar{x} + \varepsilon\omega^2 Y_1\,\bar{\Gamma}\,\bar{x} \quad \text{or} \quad [pI - Y - \varepsilon\omega^2 Y_1 \bar{\Gamma}\,(p)]\bar{x} = x_0,$$

Where *I* is a unit matrix. Let the following condition be fulfilled:

$$\det\big(pI - Y - \varepsilon\omega^2\, Y_1\,\bar{\Gamma}\,(p)\big) = p^2 + \omega^2\big(1-\varepsilon\bar{\Gamma}\,(p)\big) \neq 0, \text{ for } Re p \geq 0 \qquad (5)$$

Then multiplying the last equality from the left by the improper matrix

$\bar{\mathrm{Z}}\,(\mathrm{p}) = [pI - Y - \varepsilon\omega^2 Y_1 \bar{\Gamma}\,(p)]^{-1}$, we get the followine expression

$$\bar{x} = [pI - Y - \varepsilon\omega\,^2 Y_1\,\bar{\Gamma}\,(p)\,]^{-1} x_0 \qquad (6)$$

This formula expresses the solution to problem (4) in Laplace transforms. The function *Z(t)* is called a resolvent of the equation (4). As it follows from the expression of is representation, the resolvent *Z(t)* is a unique solution of the matrix problem.

$$Z'(t) = YZ(t) + \varepsilon\omega^2 Y_1 \int_0^1 \Gamma(t-s)Z(s)ds, Z(0) = I \qquad (7)$$

The original of solution (6) in written in the form $x(t) = Z(t)x_0$ (t≥ 0). By fulfilling condition (5), the representation of the resolvent has only a pole with negative real parts. Therefore, condition (5) is a necessary condition of uniform and asymptotic stability of the solution of problem (4) $x(t) = Z(t)x_0$ (t≥ 0). Since by inequality (4) in the half-plane $\operatorname{Re} p \geq 0$…we have the estimates

$|\varepsilon\Gamma(p)| \leq \varepsilon \int_0^\infty \Gamma(t)\, |e^{-pt}| dt \leq \varepsilon \int_0^\infty \Gamma(t)\, dt \ll 1.$ The condition (5) follows from condition (4). Thus, condition (4) provides uniform and asymptotic stability of the solution of problem (1), (2).

Problem (7) may be solved by different approximate methods. For example, $\varepsilon = 0$ this problem has matrix solution $Z(t) = e^{-Yt}$. Substituting $Z = e^{-Yt}V(t)$ in (7), we get a problem for defining we function *V(t)*

$$V'(t) = \varepsilon\omega^2 e^{-Yt} Y_1 \int_0^t \Gamma(t-s) e^{Ys} V(s) ds,\ V(0) = I \qquad (8)$$

Equation (8) shows that rate of change of the function *V(t)* in proportional to the small parameter ε, i.e. this function in slowly varying. It is said that equation (8) has a so called standard (by Lagrange) form. Therefore its solution my be sought by the averaging method. In (5) it was done in this way. However, we'll here obtain this solution by a more simple method. Suppose that Laplace transform *T(i)* is an analytic function in all it complex p-plane except isolated singular points. The inverse transformation of function (3) may be found by the following well known Bromwich (or Mellin) formula

$$T(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(pT_0 + T_1) e^{pi}}{p^2 + \omega^2 - \varepsilon\lambda^2 \bar{\Gamma}(p)} dp \qquad (9)$$

here $i = \sqrt{-1}$ and integration is conducted in the plane of complex variable *p* along an infinite straight line parallel to the axis and arranged so that all-the singular points $\bar{T}(p)$ are disposed from the left of this straight line. The integral usually is calculated by the residual theory. By this reason, it is necessary to know the poles and branching points of the integral expression that is assumed to be analytically continued to the left half of plane. The poles are the roots of the equation

$$p^2 + \omega^2 \left(1 - \varepsilon\bar{\Gamma}(p)\right) = 0 \qquad (10)$$

The left side of this equation coincides with he left side of inequality (5). If $1 - \varepsilon\bar{\Gamma}$=0 (ultimate viscosity) or $1 - \varepsilon\bar{\Gamma}$<0 (supercritical viscosity). Then equation (7) has only real roots and vibrations don't appear. For ε =0, equation (7) has two imaginary roots $p_1 = i\omega$ and $p_2 = -i\omega$ taking into account the equalities $\bar{\Gamma}(\pm i\omega) = \Gamma_c \mp i\Gamma_s$, where Γ_s and Γ_C are sin and *cos*-Fourier transforms of the kernel $\Gamma(t)$, on the left part of equation (7) we perform the following transformations:

$$p^2 + \omega^2\left(1 - \varepsilon\bar{\Gamma}(p)\right) \approx$$

$$\approx \left(p - i\omega\sqrt{1 - \varepsilon\bar{\Gamma}(i\omega)}\right)\left(p + i\omega\sqrt{1 - \varepsilon\bar{\Gamma}(-i\omega)}\right) \approx$$

$$\approx \left(p - i\omega\sqrt{1 - \varepsilon\Gamma_c} + \frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_c}} + \cdots\right)\left(p + i\omega\sqrt{1 - \varepsilon\Gamma_c} + \frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_C}} + \frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_c}} + ..\right) = 0$$

, where the remainder term of expansion is of order $O(\varepsilon^2)$. Hence we find approximate expressions of the roots of the equation (10)

$$p_{1,2} \approx -\frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_C}} \pm i\omega\sqrt{1 - \varepsilon\Gamma_C}$$

Therewith, equation (10) is verifies as follows:

$$p^2 + \omega^2\left(1 - \varepsilon\bar{\Gamma}(p)\right) = \left(p + \frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_c}}\right)^2 + \omega^2(1 - \varepsilon\Gamma_c) + O(\varepsilon^2).$$

Formula (3) accepts the form (1)

$$\bar{T}(p) \approx \frac{pT_0 + T_1}{\left(p + \frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_C}}\right)^2 + \omega^2(1 - \varepsilon\Gamma_C)}$$

Using the table of Laplace inverse transforms, we find the original

$$T(t) = e^{\frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_C}} t} \left[\begin{array}{c} T_0 \cos\omega\sqrt{1 - \varepsilon\Gamma_c}\, t + \\ \frac{T_1 - \frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_C}} T_0}{\omega\sqrt{1 - \varepsilon\Gamma_C}} \sin\omega\sqrt{1 - \varepsilon\Gamma_c}\, t \end{array} \right] \qquad (11)$$

This function describes the damping vibrations with frequency $\omega\sqrt{1 - \varepsilon\Gamma_c}$ and the damping factor $\frac{\varepsilon\omega\Gamma_s}{2\sqrt{1 - \varepsilon\Gamma_C}}$. Availability of viscous resistance of material causes damping of free vibrations amplitude by exponential law and lower frequency of there vibrations. Therewith, the damping factor is proportional to sine - Fourier transform of the kernel $\varepsilon\Gamma$, the frequency decrease to *cos* - Fourier transform of this kernel. If we linearize these expressions with respect to ε, we get

$$T(t) = e^{-\frac{\varepsilon}{2}\omega\Gamma_s t} \left[T_0 \cos\omega\left(1 - \frac{\varepsilon}{2}\Gamma_c\right) t + \frac{T_1 - \varepsilon\omega\Gamma_s T_0/2}{\omega(1 - \varepsilon\omega\Gamma_c/2)} \sin\omega\left(1 - \frac{\varepsilon}{2}\Gamma_c\right) t\right] \qquad (12)$$

This is the known approximate solution of problem (1), (2) under general form of relaxation kernel corresponding to the problem on Eigen vibrations of viscoelastic bodies and structural elements. Comparison of formula (11) and (12) shows that by linearization with respect to ε the damping factor was reduced times, vibrations frequency $(1 - \varepsilon\Gamma_c)^{-\frac{1}{2}}$ inversed $\left[1 - \frac{\varepsilon^2\Gamma_c^2}{4(1 - \varepsilon\Gamma_c/2)^2}\right]^{-\frac{1}{2}}$ times compared to formula (11). As we'll see later, by the present form, formula (11) is closer to the exact solution of problem (1), (2) than formula (12). After some mathematical operations we get the original by we solution in the form ε. Note that this solution was constructed with regard to only two complexly – conjugated poles found for any relaxation kernel *Γ(t)* .

It is obvious that depending on the form of *Γ(p)* the mother poles and branching points of the function *T(p)* may also appear. As it is seen from (13) their continuations in the sol union are concert rated at the function $B_n(t),\ n = 1,2,\ldots$. It in formula (13) we neglect all terms under sign of sun and take into attention only the terms linear with respect to ε, i.e. if we assume $\alpha = \varepsilon\omega\,\Gamma_s\,/2,\ \ \omega\,(1 - \varepsilon\Gamma_c)\,/2)$ then we get the result of A.A. Ilyushin and his [1] -colleagues' obtained by the averaging method, and also the result of the paper [2] -obtained by the method of complex modules. For $\beta = \beta_1$ and $\alpha = \alpha_1$ the frequency and damping factor correspond to the results obtained in [10]. The solution consists of two parts. The first part describes the process of damping vibration with frequency β and the damping factorα. The second part is called the transient part of the solution. This part is not vibrating and rapidly decreases as $t \to \infty$. As it is seen from the obtained results, the frequency of viscoelastic vibrations is less than the frequency of appropriate elastic vibration ω . For ω convergent to zero viscoelastic vibrations become undamped vibrations with frequency $\omega\sqrt{R(\infty)/R(0)}$ bat for sufficiently large ω viscoelastic and elastic frequencies coincide, and vibroelastic damping factor achieves is, greatest value in modulus $R'(0)/2R(0)$.

III. CONCLUSIONS

The Laplace integral transform is applied to the solution of the problem on no stationary vibrations of structural elements whose properties are described by theory of linear viscoelasticity under arbitrary hereditary kernels. The representation of solution is written rather simple, however the calculation of the Laplace inverse transform that inevitably reduces to calculation of Bromwich (Melina) integral method and residue theory is impossible if analytic dependence of hereditary kernels are not given. This situation was noted by the outstanding scientists Ilyuslin, Rabotnov and Christensen Difficulties are connected with impossibility of calculation of poles and branching points of the integrand function of Bromwich integral. The solution method of quasistatic problems of linear viscoelasticity under arbitrary dependence of hereditary kernels in time was suggested and grounded by A.A.Ilyushin. Dynamical problems of viscoelasticity are more complicated than quasistatic problems since here the reprentation of the problem irrationally depends on representation of hereditary functions. The poles of the integrand function whose real parts are the damping factor, the imaginary parts are frequencies of viscoelastic vibrations, were found by sequential approximations beginning with frequencies elastic vibrations. The convergence of the sequential approximations process was proved.

Here the parameter of the kernel ε is a small parameter. The existence of only two complexly conjugated poles was defined, the others, if they exist, may be only real negatives. The solution is obtained in the form of a series, its convergence proved, the obtained of the remainder tern of the series in reduced. It is shown that for sufficiently small values of time the frequencies of viscoelastic and elastic vibrations coincide ,the damping factor has its greatest value .The frequency of viscoelastic vibrations is smaller than the appropriate frequency of elastic vibrations by the quantity γ, moreover, the first term of the series γ with respect to degrees of the parameter of the kernel ε is proportional to cosine Fourier transform of relaxation kernel α, and the first term of the series of the damping factor a with respect to degrees of the parameter of the kernel ε is proportional to the sine Fourier transform of relaxation kernel. If in these formulae we neglect all degrees of ε higher than one, then we get the results of solution of the appropriate problem on ligen vibrations of viscoelastic bodies and structural elements, obtained by A.A.Ilyushin, and his collages by the averaging method [1]. For sufficiently large values of time, a viscoelastic material behaves as elastic one, i.e. the damping factor decreases converging to zero, and frequency of vibrations decrease and tends to its long-term value in which momentary modulus are replaced by long-term (relaxed) elasticity modulus.

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