CONSTRUCTION OF INTEGER PERIODIC FUNCTIONS ACCORDING TO GIVEN VALUES AT GIVEN POINTS

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From the general theory of integer functions, it is well-known that many properties of them directly depend on the corresponding sequence properties of these functions' zeros. For instance, an indicator of the convergence of integer functions' zero sequence does not exceed the order of this function; the upper density of zero sequence does not exceed its type, etc. [1]. These properties lie in the basis of construction of integer functions according to given values at given points.

Let's note that in the case of integer periodic functions (with the period of 2π), it is sufficient to set the points at the band $\prod = \{z \in C : 0 \le \operatorname{Re}(z) < 2\pi\}$ (please refer to [2], [3]). Then, there occurs a natural problem of learning the properties of sequence of the following type: $\Lambda = \{\lambda_\kappa\}, \lambda_\kappa \in \prod, \kappa = 1,2,...$ and a set of the types $\{\Lambda + 2\pi n\}$, $n = 0, \pm 1, \pm 2, \ldots$

The present paper is dedicated to this problem.

Sets Λ and Ω.

Let's consider a sequence of complex numbers

$$\Lambda = \{\lambda_\kappa\} : \kappa = 1,2, \ \ldots, 0 \le \operatorname{Re}(\lambda_k) < 2\pi, |\lambda_1| \le |\lambda_2| \le \ \ldots, \ \lim_{k\to\infty} |\lambda_k| = \infty .$$

Let the number $\tau \ge 0$ be an indicator of convergence of the sequence Λ, i.e.

$$\tau = \overline{\lim_{r\to\infty}} \frac{\ln(n(r))}{\ln(r)},$$

where $n(r)$ is a number of members of the sequence Λ, fallen into the circle $|z| \le r$.

Let's indicate the upper density of the sequence Λ through Δ,

$$\Delta = \overline{\lim_{r\to\infty}} \frac{n(r)}{r^\tau}.$$

Along with this sequence, let's consider the set $\Omega = \{\Lambda + 2\pi n\}$, $n = 0, \pm 1, \pm 2, \ \ldots$ Let's indicate a convergence indicator through q, and the upper density of this set through Δ_1. Therefore

$$q = \overline{\lim_{r\to\infty}} \frac{\ln(N(r))}{\ln(r)}, \ \Delta_1 = \overline{\lim_{r\to\infty}} \frac{N(r)}{r^q},$$

where $N(r)$ is a number of members of the set Ω fallen into the circle $|z| \le r$. Then the following theorem is true:

Theorem 1. The numbers τ and q are connected by the ratio of $q = \tau + 1$.

Proof. Let's consider the circle $|z| \le r$. Let $n(r)$ be a number of members of the sequence Λ, and $\mathrm{N}(r)$ be a number of members of the set Ω fallen into the circle $|z| \le r$.

Let $[\mathrm{r}/2\pi] = m$. Then, $2\pi m \le r \le 2\pi(m+1)$. It's easy to see that the following inequality is valid:

$$\mathrm{N}(\mathrm{r}) \le 2n(r)2\pi(m+1) = 2n(r)2\pi m + 2n(r)2\pi \le 2n(r)r + 2n(r)2\pi .$$

Let $\mathrm{R} = \mathrm{r}/\sqrt{2}$ and $[\mathrm{R}/2\pi] = l$. It is obvious that

$$\mathrm{N}(r) \ge 2n(R)2\pi l \ge 2n(R)R .$$

Further, having united the values for $\mathrm{N}(r)$, we get

$$2\mathrm{n}(\mathrm{R})R \le \mathrm{N}(r) \le 2n(r)(r+2\pi). \tag{1.1}$$

It is obvious that

$$\frac{\ln(2n(R)R)}{\ln(r)} \le \frac{\ln(\mathrm{N}(r))}{\ln(r)} \le \frac{\ln(2n(r)(r+2\pi))}{\ln(r)},$$

$$\frac{\ln(2)+\ln(n(R))+\ln(R)}{\ln(\sqrt{2})+\ln(R)} \le \frac{\ln(\mathrm{N}(r))}{\ln(r)} \le \frac{\ln(2\ \ )+\ln(n(r))+\ln(r+2\pi)}{\ln(r)}.$$

If we move to the limit with $\mathrm{r} \to \infty$ in the last inequalities, then we can get the following desired ratio

$$\tau + 1 \le \mathrm{q} \le \tau + 1 .$$

The Theorem 1 is proved.

The following statement shows a relation between the upper density of the sequence Λ and the upper density of the set Ω.

Theorem 2. The numbers Δ and Δ_1 satisfy the following inequalities

$$\left(\sqrt{2}\right)^{1-\tau} \Delta \le \Delta_1 \le 2\Delta .$$

Proof. Let $\mathrm{n}(\mathrm{r})$ be a number of members of the sequence Λ and $\mathrm{N}(r)$ - a number of members of the set Ω fallen into the circle $|z| \le r$, as in the Proof of the Theorem 1. The numbers $\mathrm{n}(\mathrm{r})$ and $\mathrm{N}(r)$ satisfy the inequalities (1.1). Having divided all the parts (1.1) to r^{q}, we get the following:

$$\frac{2\mathrm{n}(\mathrm{R})R}{r^q} \le \frac{\mathrm{N}(r)}{r^q} \le \frac{2n(r)(r+2\pi)}{r^q},$$

$$\frac{n(R)}{\left(\sqrt{2}\right)^{q-2} R^{\tau}} \le \frac{\mathrm{N}(r)}{r^q} \le \frac{2n(r)}{r^{\tau}} + \frac{4\pi n(r)}{r^q}.$$

Then, moving to the limit with $r \to \infty$ in the last inequalities, we get the following desired inequalities.

Note. From the Theorem 2 it follows that the numbers Δ and Δ_1 either turn into zero at the same time, equal to infinity or are finite number.

Example. Let $\Lambda = \{\mathrm{i}k\}$, $k = 0, \pm 1, \pm 2, \ldots$. Then $\Omega = \{\mathrm{i}k + 2\pi n\}$, $k, n = 0, \pm 1, \pm 2, \ldots$ It is obvious that $\tau = 1$, $\Delta = 2$. According to the Theorem 1 and 2, we get $q = 2$ and $2 \le \Delta_1 \le 4$.

Let's consider the sequence $\Lambda' = \{\mathrm{Im}(\lambda_k)\}$ along with the sequence $\Lambda = \{\lambda_k\}$. Let the number τ' be an indicator of the convergence of the sequence Λ' and the number Δ' be the upper density of this sequence.

Suggestion. The sequences Λ and Λ' have the following property: $\tau = \tau', \Delta = \Delta'$.

Indeed, according to the definition

$$\tau = \overline{\lim_{k\to\infty}} \frac{\ln k}{\ln|\lambda_k|}.$$

But on the other hand,

$$|\mathrm{Im}(\lambda_k)| \le |\lambda_k| \le |\mathrm{Im}(\lambda_k)| + 2\pi .$$

Therefore,

$$\tau = \overline{\lim_{k\to\infty}} \frac{\ln k}{\ln|\lambda_k|} = \overline{\lim_{k\to\infty}} \frac{\ln k}{\ln|\mathrm{Im}(\lambda_k)|} = \tau'$$

For the upper density we also get

$$\Delta = \overline{\lim_{k\to\infty}} \frac{k}{|\lambda_k|^{\tau}} = \overline{\lim_{k\to\infty}} \frac{k}{|\mathrm{Im}(\lambda_k)|^{\tau}} = \Delta'$$

Therefore, the convergence indicator and the upper density of the sequence Λ can be calculated through the following formulae

$$\tau = \overline{\lim_{k\to\infty}} \frac{\ln k}{\ln|\mathrm{Im}(\lambda_k)|}, \quad \Delta = \overline{\lim_{k\to\infty}} \frac{k}{|\mathrm{Im}(\lambda_k)|^{\tau}},$$

Theorem 3. The sequence $\{s_l\}$, where

$$s_l = \sum_{n=-l}^{l} \frac{1}{(\lambda_1 + 2\pi n)^m},$$

converges for every integer $m \ge 1$.

Proof. Let $m = 1$. For every integer $n > 1$, we have

$$\left| \frac{1}{(\lambda_1 + 2\pi n)} + \frac{1}{\lambda_1 - 2\pi n} \right| = \frac{2|\lambda_1|}{|\lambda_1^2 - (2\pi n)^2|}.$$

Further, we get

$$\left|\lambda_1^2-(2\pi n)^2\right|=\left(\left((\mathrm{Re}(\lambda_1))^2-(\mathrm{I}m(\lambda_1))^2-(2\pi n)^2\right)^2+(2\,\mathrm{Re}(\lambda_1)\mathrm{I}m(\lambda_1))^2\right)^{1/2}\geq$$

$$\geq\left(\left((2\pi)^2-(\mathrm{I}m(\lambda_1))^2-(2\pi n)^2\right)^2+0\right)^{1/2}=\left|\,|\mathrm{I}m(\lambda_1)|^2+(2\pi)^2(n^2-1)\right|$$

Therefore,

$$|s_l|\leq\frac{1}{|\lambda_1|}+\frac{2|\lambda_1|}{\left|\lambda_1^2-(2\pi)^2\right|}+2|\lambda_1|\sum_{n=2}^{l}\frac{1}{(2\pi)^2(n^2-1)+|\mathrm{I}m(\lambda_1)|^2}.$$

It is obvious that the limit of the latter exists and is finite.

Now, let $m>1$. In this case, the validity of the Lemma caused by the evidence showing the series

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\lambda_1+2\pi n)^m}$$

absolutely converges. The Theorem 3 is proved. The following Lemma is valid:

Lemma 1. The series

$$\sum_{k=1}^{\infty}\upsilon_k,\ \text{где}\ \ \upsilon_k=\sum_{n=-\infty}^{\infty}\frac{1}{(\lambda_k+2\pi n)^m},$$

converge for every integer $m>1$.

References

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