ON THE SPECTRUM OF A CLASS OF A GENERALIZED DIFFERENCE OPERATOR OVER THE SPACE l_p, $p \ge 1$

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The main purpose of this paper is to determine the spectrum of the generalized difference operator Δ_a, over the sequence space l_p, $p \ge 1$. The results of this paper generalize the corresponding results of [1] and [3].

1. Introduction, Preliminaries, Background and Notation

By $B(X)$, we denote the set of all bounded linear operators on the Banach space X *into* itself. Let $X \ne \varnothing$ be a complex normed space and consider a linear operator $T : D(T) \to X$, with domain $D(T) \subseteq X$. With T we associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. By a regular value λ of T we mean a complex number such that
(**R1**) T_λ^{-1} exists,
(**R2**) T_λ^{-1} is bounded,
(**R3**) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its compliment $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the spectrum of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point* (*discrete*) *spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies (**R3**) but not (**R2**), that is T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but not satisfy (**R3**), that is the domain of T_λ^{-1} is not dense in X.

By w, we shall denote the space of all real valued sequences. Any vector subspace of w is called a sequence space. We write l_∞ for the spaces of all bounded sequences. Also by l_1 and l_p, we denote the space of all absolutely and p-absolutely convergent series, respectively.

P. Srivastava and S. Kumar [8] introduced the generalized difference operator Δ_a on the sequence space c_0 as follows:

$\Delta_a : c_o \to c_o$ is defined by

$$\Delta_a x = \Delta_a(x_n) = (a_n x_n - a_{n-1} x_{n-1})_{n=0}^{\infty} \text{ with } x_{-1} = 0,$$

where (a_k) is either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty} a_k = a > 0 \text{ and } a_0 \le 2a. \tag{1.1}$$

P. Srivastava and S. Kumar determined the spectrum and fine spectrum of the operator Δ_a over the sequence space c_0 in [8]. The same problem, in the case when the sequence (a_k) is assumed to be constant except for finitely many elements was investigated in [3].

In this paper we determine the spectrum of the generalized difference operator Δ_a on the sequence space $l_p,\ p \ge 1$. The results of our paper not only generalize the corresponding results of [1] and [3] but also give results for some more operators.

We summarize the knowledge in the existing literature concerning with the spectrum of the linear operator defined by some particular limitation matrices over some sequence spaces. The fine spectrum of the difference operator Δ over the sequence space l_p, $(p \ge 1)$ is determined by A. Akhmedov and F. Başar [1] and over the sequence space c_0 and c by B. Altay and F. Başar [4]. B. De Malafosse [7] computed the spectrum of the difference operator on the space s_r. A. Akhmedov and F. Başar [2] determined the fine spectrum of the difference operator on the space bv_p, $(1 \le p < \infty)$. Note that the sequence space bv_p was introduced and studied by B. Altay and F. Başar [5]. The continuous dual of bv_p determined by A. Akhmedov in [2].

2. The spectrum of the operator Δ_a on the sequence space $l_p,\ p \ge 1$

In this section, we compute the spectrum and the point spectrum of the operator Δ_a on the sequence space $l_p,\ p \ge 1$. Throughout this paper, the sequence (a_k) satisfies conditions (1.1).

Theorem 1. $\Delta_a \in B(l_p)$ with a norm satisfies $2^{\frac{1}{p}} a_0 \le \|\Delta_a\|_{l_p} \le 2a_0$.

Proof. Proof is simple. So we omit it. □

The spectrum of the operator Δ_a on the space $l_p,\ p \ge 1$ is given by the following theorem.

Theorem 2. $\sigma(\Delta_a, l_p) = \{\lambda \in \mathbb{C} : |\lambda - a| \le a\}$.

Proof. Let $\lambda \notin \{\lambda \in \mathbb{C} : |\lambda - a| \le a\}$ and let $y = (y_k) \in l_1$. Then $|\lambda - a| > a$. By solving the equation $(\Delta_a - \lambda I)x = y$, for $x = (x_k)$ in terms of y, we get

$$x_k = \frac{a_0 a_1 \ldots a_{k-1}}{(a_0-\lambda)(a_1-\lambda)\ldots(a_k-\lambda)} y_0 + \ldots + \frac{a_{k-1}}{(a_{k-1}-\lambda)(a_k-\lambda)} y_{k-1} + \frac{1}{(a_k-\lambda)} y_k \qquad ,k \in \mathbb{N}$$

Then, $\sum_k |x_k| \le \sum_k R_k |y_k|$,where

$$R_k = \frac{1}{|a_k-\lambda|} + \frac{a_k}{|a_k-\lambda||a_{k+1}-\lambda|} + \frac{a_k a_{k+1}}{|a_k-\lambda||a_{k+1}-\lambda||a_{k+2}-\lambda|} + \ldots. \qquad ,k \in \mathbb{N}.$$

Let

$$R_{n,k} = \frac{1}{|a_k-\lambda|} + \frac{a_k}{|a_k-\lambda||a_{k+1}-\lambda|} + \frac{a_k a_{k+1}}{|a_k-\lambda||a_{k+1}-\lambda||a_{k+2}-\lambda|} + \ldots + \frac{a_k a_{k+1} \ldots a_{k+n}}{|a_k-\lambda||a_{k+1}-\lambda||a_{k+n+1}-\lambda|} \qquad ,k,n \in \mathbb{N}.$$

Then $R_n = \lim_{k\to\infty} R_{n,k} = \frac{1}{|a-\lambda|} + \frac{a}{|a-\lambda|^2} + \frac{a^2}{|a-\lambda|^3} + \ldots + \frac{a^{n+1}}{|a-\lambda|^{n+2}}$, and so,

$$R = \lim_{n\to\infty} R_n = \frac{1}{|a-\lambda|}\left(1+\frac{a}{|a-\lambda|}+\frac{a^2}{|a-\lambda|^2}+\frac{a^3}{|a-\lambda|^3}+...\right) < \infty ,$$

since $\frac{a}{|a-\lambda|} < 1$. Then (R_k) is a sequence of positive real numbers which has a limit *R*. Then (R_k) is bounded and so $\sup_k R_k < \infty$. Thus, $\sum_k |x_k| \le \sum_k R_k |y_k| \le \sup_k R_k \sum_k |y_k| < \infty,$ since $y \in l_1$. This shows that $(\Delta_a - \lambda I)^{-1} \in (l_1; l_1)$.

Similarly, let $y = (y_k) \in l_\infty$. By solving the equation $(\Delta_a - \lambda I)x = y$, for $x = (x_k)$ in terms of *y*, we get

$$|x_k| \le \left(\frac{1}{|a_k-\lambda|}+\frac{a_{k-1}}{|a_{k-1}-\lambda||a_k-\lambda|}+\frac{a_{k-1}a_{k-2}}{|a_{k-2}-\lambda||a_{k-1}-\lambda||a_k-\lambda|}+...+\frac{a_0a_1...a_{k-1}}{|a_0-\lambda||a_1-\lambda|...|a_k-\lambda|}\right)\sup_k|y_k|.$$

Let $S_n = \frac{1}{|a_n-\lambda|}+\frac{a_{n-1}}{|a_{n-1}-\lambda||a_n-\lambda|}+\frac{a_{n-1}a_{n-2}}{|a_{n-2}-\lambda||a_{n-1}-\lambda||a_n-\lambda|}+...+\frac{a_0a_1...a_{n-1}}{|a_0-\lambda||a_1-\lambda|...|a_n-\lambda|}$, for each $n \in \mathbb{N}$. Clearly, for each $n \in \mathbb{N}$, the series S_n is convergent since it is finite. Next we prove that $\sup_k S_n$ is finite. Since $\lim_{n\to\infty}\frac{a_n}{|a_n-\lambda|} = \frac{a}{|a-\lambda|} = q < 1,$ then there exists $k \in \mathbb{N}$ such that $\frac{a_n}{|a_n-\lambda|} < q_0 < 1$, for all $n \ge k+1$ and so we get

$$S_{n+k} \le \left(\frac{a_0a_1...a_k}{|a_0-\lambda||a_1-\lambda|...|a_k-\lambda|}q_0^{\,n-1}+\frac{a_1a_2...a_k}{|a_1-\lambda||a_2-\lambda|...|a_k-\lambda|}q_0^{\,n-2}+...+q_0+1\right)\frac{1}{|a_{n+k}-\lambda|}.$$

If we put, for each $i = 0,1,2,...,k,$ $m_i = \frac{a_ia_{i+1}...a_k}{|a_i-\lambda||a_{i+1}-\lambda|...|a_k-\lambda|}$, and $A = \max_{1\le i\le k}\{m_i\}$, then we

$$S_{n+k} \le A\left(1+q_0+q_0^{\,2}+q_0^{\,3}+...+q_0^{\,n-1}\right)\frac{1}{|a_{n+k}-\lambda|} \le A\left(1+q_0+q_0^{\,2}+q_0^{\,3}+...\right)\frac{1}{|a_{n+k}-\lambda|}.$$

But, for large *n*, we have $\frac{1}{|a_{n+k}-\lambda|} < q_1 < \frac{1}{a}$, and so $S_{n+k} \le \frac{Aq_1}{1-q_0},$ for all $n \ge k+1$. Thus, $\sup_k S_k < \infty$. This shows that $|x_k| \le \sup_k S_k \sup_k |y_k| < \infty,$ since $y \in l_\infty$. This shows that $(\Delta_a - \lambda I)^{-1} \in (l_\infty; l_\infty)$.

Therefore $(\Delta_a - \lambda I)^{-1} \in (l_1; l_1) \cap (l_\infty; l_\infty)$, and so, $(\Delta_a - \lambda I)^{-1} \in (l_p; l_p)$ [see [6]Theorem (9), pp. 174]. Thus $\sigma(\Delta_a, l_p) \subseteq \{\lambda \in \mathbb{C} : |\lambda - a| \le a\}$.

Conversely, let $\lambda \in \mathbb{C}$ with $|\lambda - a| < a$. If we compute $(\Delta_a - \lambda I)^{-1}e_1$, where e_1 is the unit sequence (1, 0, 0, 0, …), then we can easily see that $(\Delta_a - \lambda I)^{-1}e_1 \notin l_p$, and so $(\Delta_a - \lambda I)^{-1} \notin B(l_p)$. Then $\lambda \in \sigma(\Delta_a, l_p)$. Therefore $\{\lambda \in \mathbb{C} : |\lambda - a| < a\} \subseteq \sigma(\Delta_a, l_p)$. But, $\sigma(\Delta_a, l_p)$ is a compact set, and so it is closed. Then $\{\lambda \in \mathbb{C} : |\lambda - a| \le a\} \subseteq \sigma(\Delta_a, l_p)$. This completes the Proof. □

From Theorem 1 and Theorem 2, we can easily prove that $2a \le \|\Delta_a\|_{l_p} \le 2a_0$.

The point spectrum of the operator Δ_a is given by the following theorem

Theorem 3. $\sigma_p(\Delta_a, l_p) = \varnothing,\ p \ge 1$

Proof. Consider the equation $\Delta_a x = \lambda x$ for $x \neq \theta = (0,0,0,...)$ in l_p. Then

$$(a_0 - \lambda)x_0 = 0 \text{ and } (a_k - \lambda)x_k - a_{k-1}x_{k-1} = 0, \text{ for all } k = 1,2,3,\dots .$$

Hence, for all $\lambda \notin \{a_k : k \in \mathbb{N}\}$, we have $x_k = 0$, for all $k \in \mathbb{N}$, which is a contradiction. So, $\lambda \notin \sigma_p(\Delta_a, l_p)$. This shows that $\sigma_p(\Delta_a, l_p) \subseteq \{a_k : k \in \mathbb{N}\}$.

Now, if $\lambda = a_i$ and there exists $j \in \mathbb{N}$, $j > i$ such that $a_i = a_j$, then we can easily see that $x_k = 0$ for all $k < j$. Then we have the following cases:

Case (i): Let (a_k) is such that $a_i \neq a_j$ for all $i, j \in \mathbb{N}$ and let, $\lambda = a_0$. If $x_0 = 0$, then $x_k = 0$ for all $k \in \mathbb{N}$ and so we have a contradiction as $x \neq \theta$. Also, if $x_0 \neq 0$ then $x_{k+1} = \dfrac{a_k}{a_{k+1} - a_0} x_k$ for all $k = 0,1,2, \dots$, and hence

$$\left|\frac{x_{k+1}}{x_k}\right| = \left|\frac{a_k}{a_0 - a_k}\right| > 1, \text{ for all } k \geq 1 .$$

Then, $x \notin l_p$. Similarly, we can prove that $a_j \notin \sigma_p(\Delta_a, l_p)$, for all $j \in \mathbb{N}$. Thus $\sigma_p(\Delta_a, l_p) = \varnothing$ in this case.

Case (ii): If (a_k) is such that there exists $m \in \mathbb{N}$ with $a_i \neq a_j$ for all $i, j \geq m$, then we can prove, as in Case (i), that $a_j \notin \sigma_p(\Delta_a, l_p)$, for all $j \in \mathbb{N}$. Thus $\sigma_p(\Delta_a, l_p) = \varnothing$.

Case (iii): If (a_k) is not as in Case (i) or Case (ii), that is for all $m \in \mathbb{N}$ there exists $i < m$ and $j \geq m$ such that $a_i = a_j$, then we must have $x = \theta$ which is a contradiction. Thus $\sigma_p(\Delta_a, l_p) = \varnothing$ in this case.

This completes the proof. □

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