

## CONSTRUCTION OF A SECOND DERIVATIVE ONE-STEP METHOD OF FOURTH ORDER ACCURACY AND ITS APPLICATION TO THE SOLUTION OF VOLTERRA INTEGRAL EQUATION OF SECOND KIND

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There exists a wide class of approximate methods for solving Volterra integral equations. However a numerical method that could provide reliability of the obtained results or regularity of the volume of calculation works at each integration step has not been built up to now. Therefore more often there arises a necessity to construct numerical methods having wide range of stability and possessing some properties of the solution of the initial problem. Unlike the known papers, here we suggest a specific method that provides regularity of the number of inversions to calculation of the integral's kernel.

### Introduction.

Consider the following second kind Volterra integral equation:

$$y(x) = f(x) + \int_{x_0}^x k(x, s, y(s)) ds, \quad x_0 \leq x \leq X \quad (1)$$

that sometimes is called Volter-Uruson type integral equation.

Investigation of variable boundary integral equations begins with Abel's known paper. Since then many scientists are engaged in finding approximate solutions of equation (1) taking into account that to find the exact solution of equation (1) is not always succeeded even in the linear case.

Recently the numerical solution of equation (1) is mostly investigated. Usually in constructing a numerical method for solving equation (1) mainly the quadrature methods are used. However, after applying the quadrature formula to the solution of equation (1), we get a variable boundary integral sum. While using this sum, the volume of computational works increases when passing from the moving partition point to the next one. Here we construct one specific method that provides constant number of calculations of the integral's kernel at each step.

Assume that equation (1) has a unique continuous solution determined on the segment  $[x_0, X]$ . For finding the numerical solution to equation (1), we partition the segment  $[x_0, X]$  into  $N$  equal parts by means of the constant step  $h > 0$  and determine the partitioning points in the form:  $x_m = x_0 + mh$  ( $m = 0, 1, 2, \dots, N$ ). Denote the approximate values of the solution of equation (1) at the points  $x_m$  by  $y_m$ , the exact values by  $y(x_m)$ . Consider the construction of the numerical method for solving equation (1).

### § 1. Construction of the fourth degree one-step method.

Many papers have been devoted to the numerical solution of a nonlinear integral equation of type (1). In these papers the quadratures method or its modifications (see [1]) are mostly investigated. Showing some deficiencies of the quadratures method, the multi-step methods with constant coefficients for the solution of equation (1) were constructed and investigated in [2]. However in all these papers, those methods are used in which only calculation of the integral's kernel are investigated. Here we suggest a method where calculation of the integral's kernel and its first partial derivatives is used. It is easy to see that if the integral's kernel, the function  $k(x, y, z)$ , is independent on  $x$ , the finding of the solution of equation (1) will be equivalent to finding of the Cauchy problem for first order ordinary differential equations. Therefore, the method constructed in this report is called a second derivative method. To this

end, apply the quadrature method to the solution of equation (1) and write it at the point  $x_{n+1}$ . Then, we have:

$$y(x_{n+1}) = f_{n+1} + \int_{x_0}^{x_{n+1}} k(x_{n+1}, s, y(s)) ds, \quad (1.1)$$

where  $f_m = f(x_m)$  ( $m = 0, 1, \dots, N$ ).

Consider the difference  $y(x_{n+1}) - y(x_n)$ . Then we get:

$$y(x_{n+1}) - y(x_n) = f_{n+1} - f_n + \int_{x_0}^{x_{n+1}} k(x_{n+1}, s, y(s)) ds - \int_{x_0}^{x_n} k(x_n, s, y(s)) ds.$$

Applying the Lagrange theorem to the difference

$$k(x_{n+1}, z, y) - k(x_n, z, y) = hk'_x(\xi_{n+1}, z, y), \quad x_n < \xi_{n+1} < x_{n+1}$$

and taking into account

$$\int_{x_0}^{x_n} (k(x_{n+1}, s, y(s)) - k(x_n, s, y(s))) ds,$$

we get

$$y(x_{n+1}) - y(x_n) = f_{n+1} - f_n + \int_{x_n}^{x_{n+1}} k(x_{n+1}, s, y(s)) ds + h \int_{x_0}^{x_n} k'_x(\xi_{n+1}, s, y(s)) ds. \quad (1.2)$$

Consider the calculation of the integral  $v_n = \int_{x_0}^{x_n} k'_x(\xi_{n+1}, s, y(s)) ds$ .

Suppose that we found the solution of equation (1) by any method. Then, taking into account the obtained solution in equation (1), we get an identity wherefrom we can get:

$$y'(x) = f'(x) + k(x, x, y(x)) + \int_{x_0}^x k'_x(x, s, y(s)) ds.$$

Here assuming  $x = \xi_{n+1}$ , we get:

$$\int_{x_0}^{\xi_{n+1}} k'_x(\xi_{n+1}, s, y(s)) ds = y'(\xi_{n+1}) - f'(\xi_{n+1}) - k(\xi_{n+1}, \xi_{n+1}, y(\xi_{n+1})).$$

Allowing for the obtained one for calculating the quantity  $v_n$ , we get the following:

$$v_n = y'(\xi_{n+1}) - f'(\xi_{n+1}) - k(\xi_{n+1}, \xi_{n+1}, y(\xi_{n+1})) - \int_{x_n}^{\xi_{n+1}} k'_x(\xi_{n+1}, s, y(s)) ds. \quad (1.3)$$

It is easy to show that

$$\begin{aligned} hv_n + \int_{x_n}^{x_{n+1}} k(x_{n+1}, s, y(s)) ds &= hy'(\xi_{n+1}) - hf'(\xi_{n+1}) - hk(\xi_{n+1}, \xi_{n+1}, y(\xi_{n+1})) + \\ &+ h \int_{x_n}^{\xi_{n+1}} k(x_n, s, y(s)) ds + \int_{\xi_{n+1}}^{x_{n+1}} k(x_{n+1}, s, y(s)) ds. \end{aligned}$$

Taking into account the one obtained in (1.2), we have:

$$y(x_{n+1}) - y(x_n) - hy'(\xi_{n+1}) = f(x_{n+1}) - f(x_n) - hf'(\xi_{n+1}) - hk(\xi_{n+1}, \xi_{n+1}, y(\xi_{n+1})) + \\ + h \int_{x_n}^{\xi_{n+1}} k(x_n, s, y(s)) ds + \int_{\xi_{n+1}}^{x_{n+1}} k(x_{n+1}, s, y(s)) ds. \quad (1.4)$$

Obviously,

$$y(x_{n+1}) - y(x_n) - hy'(\xi_{n+1}) \neq 0,$$

since the choice of the quantity  $\xi_{n+1}$  depends on the function  $k(x, y, z)$ . Therefore, using the following difference relation

$$hy'(\xi_{n+1}) \approx \sum_{i=0}^m \alpha_i y_{n+i+1},$$

we can write:

$$y(x_{n+m}) - y(x_{n+m+1}) - hy'(\xi_{n+m}) \approx \sum_{i=0}^m \alpha_i y_{n+i}.$$

If we take into account sufficient smallness of  $\xi_{n+1}$  we can write

$$k(\xi_{n+m}, \xi_{n+m}, y(\xi_{n+m})) \approx \sum_{i=0}^m \beta_i k(x_{n+i}, x_{n+i}, y(x_{n+i})).$$

If for calculating the integrals in equation (1.4) we use Hermitian interpolational polynomials and take into account the above stated one, we can rewrite the obtained formula in the following form:

$$\sum_{i=0}^m \alpha_i y_{n+i} = \sum_{i=0}^m \alpha_i f_{n+i} + h \sum_{j=0}^m \sum_{i=0}^m \beta_i^{(j)} k(x_{n+j}, x_{n+i}, y_{n+i}) + h^2 \sum_{j=0}^m \sum_{i=0}^m \gamma_i^{(j)} g(x_{n+j}, x_{n+i}, y_{n+i}), \quad (1.5)$$

where the coefficients  $\alpha_i, \beta_i^{(j)}, \gamma_i^{(j)}$  ( $i, j = 0, 1, \dots, m$ ) are some real numbers, that  $\alpha_m \neq 0$ , and function  $g(x, y, z) = k'_z(x, y, z) + k'_y(x, y, z)y'$ ,  $f_v = f(x_v)$  ( $v = 0, 1, 2, \dots$ ).

Consider finding of the coefficients  $\alpha_i, \beta_i^{(j)}, \gamma_i^{(j)}$  ( $i, j = 0, 1, \dots, m$ ).

Obviously, if method (1.5) has a definite accuracy, it retains this accuracy also in the case when the integral's kernel the function  $k(x, z, y)$  is independent of  $x$ , i.e.  $k(x, z, y) \equiv F(z, y)$ . Therefore into equation (1) we put  $k(x, s, z) \equiv F(s, y)$ .

Then from equation (1) we have:

$$y(x) = f(x) + \int_{x_0}^x F(s, y(s)) ds.$$

Hence we get

$$y' = f'(x) + F(x, y), \quad y(x_0) = f(x_0), \quad (1.6)$$

that is the Cauchy problem for first order ordinary differential equations. While applying method (1.5) to the solution of problem (1.6), the method (1.5) has the form:

$$\sum_{i=0}^m \alpha_i y_{n+i} = \sum_{i=0}^m \alpha_i f_{n+i} + h \sum_{i=0}^m \beta_i F_{n+i} + h^2 \sum_{i=0}^m \gamma_i H_{n+i} \quad (n = 0, 1, \dots, N - m), \quad (1.7)$$

where  $\beta_i = \sum_{j=0}^m \beta_i^{(j)}$ ,  $\gamma_i = \sum_{j=0}^m \gamma_i^{(j)}$  ( $i = 0, 1, \dots, m$ ),  $F_v = F(x_v, y_v)$ ,  $H_v = H(x_v, y_v)$ ,

$$H(x, y) = F'_x(x, y) + F'_y(x, y)F(x, y).$$

For finding the coefficients  $\alpha_i, \beta_i, \gamma_i$  ( $i = 0, 1, \dots, m$ ), we suppose that the method (1.7) has degree  $p$ , and consequently for sufficiently smooth function  $z(x)$  we can write:

$$\sum_{i=0}^m (\alpha_i z(x+ih) - h\beta_i z'(x+ih) - h^2 \gamma_i z''(x+ih)) = O(h^{p+1}), \quad h \rightarrow 0. \quad (1.8)$$

It is known that in order (1.8) hold, the unknowns  $\alpha_i, \beta_i, \gamma_i$  ( $i = 0, 1, \dots, m$ ) should satisfy the following conditions:

$$\sum_{i=0}^k \alpha_i = 0; \quad \sum_{i=0}^k i\alpha_i = \sum_{i=0}^k \beta_i, \dots, \quad (1.9)$$

$$\sum_{i=0}^k \frac{i^l}{l!} \alpha_i = \sum_{i=0}^k \frac{i^{l-1}}{(l-1)!} \beta_i + \sum_{i=0}^k \frac{i^{l-2}}{(l-2)!} \gamma_i \quad (l = 2, 3, \dots, p; 0! = 1).$$

Consider the case  $k = 1$ . Without losing generality we can assume  $\alpha_1 = 1$ , and get that  $\alpha_0 = -1$ . In this case from (1.9) we have:

$$\beta_1 + \beta_0 = 1, \quad \beta_1 + \gamma_0 + \gamma_1 = \frac{1}{2}, \quad \beta_1 + 2\gamma_1 = \frac{1}{3}, \quad \beta_1 + 3\gamma_1 = \frac{1}{4}. \quad (1.10)$$

Solving this system, we get  $\beta_0 = \beta_1 = 1/12$ . Notice that system (1.10) has a unique solution. Therefore the obtained method of type (1.7) with power  $p = 4$  is unique. However in this case the constructed method of type (1.5) is not unique. Cite some of them:

$$y_{n+1} = y_n + f_{n+1} - f_n + h(k(x_n, x_n, y_n) + k(x_n, x_{n+1}, y_{n+1})) + h^2(2g(x_n, x_n, y_n) - g(x_{n+1}, x_{n+1}, y_{n+1}) - g(x_n, x_{n+1}, y_{n+1}))/24,$$

$$y_{n+1} = y_n + f_{n+1} - f_n + h(k(x_n, x_n, y_n) - k(x_n, x_{n+1}, y_{n+1})) + 2k(x_{n+1}, x_{n+1}, y_{n+1})/2 - h^2(g(x_n, x_n, y_n) - 2g(x_{n+1}, x_{n+1}, y_{n+1}) + g(x_n, x_{n+1}, y_{n+1}))/12.$$

Notice that if we consider the following finite-difference method

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i y'_{n+i} + h^2 \sum_{i=0}^k \gamma_i y''_{n+i},$$

and take into account  $y'(x) = f'(x) + F(x, y)$ ,  $y''(x) = f''(x) + g(x, y)$ , we have

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f'_{n+i} + h^2 \sum_{i=0}^k \gamma_i f''_{n+i} + h \sum_{i=0}^k \beta_i F_{n+i} + h^2 \sum_{i=0}^k \gamma_i g_{n+i}.$$

Taking into account that the power of the method ( ) equals  $p$ , we can write

$$\sum_{i=0}^k \alpha_i y_{n+i} = h \sum_{i=0}^k \beta_i f'_{n+i} + h^2 \sum_{i=0}^k \gamma_i f''_{n+i}.$$

After considering the obtained one in ( ), we get

For calculating  $y'_{n+1}$  we can use the method from [2].

## References

1. A.F. Verlan, V.S. Sizikov. Integral equations: methods, algorithms, programs. (In Russian) Naukova Dumka, Kiev (1986), 544 p.
2. G.Yu.Mehdiyeva, M.N. Imanova. On an application of the finite difference method (In Russian). News of Baku University, Series of physico-math.sciences, 2008, №2, pp.73-78.