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METHOD OF SOLVING MIXED-INTEGER PROGRAMMING WITH ONLY ONE LIMIT

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Let's consider the following problem:

$$\sum_{j=1}^{N} c_j x_j \to \max \tag{1}$$

$$\sum_{j=1}^{N} a_j x_j \le b \tag{2}$$

$$0 \le x_j \le d_j \quad , \ (j = \overline{1, N}) \tag{3}$$

$$x_j,\ d_j \text{ - the whole numbers, } (j = \overline{1, n}),\ (n \le N) \tag{4}$$

Here $c_j > 0$, $a_j > 0$, $d_j > 0$ ($j = \overline{1, N}$), $b > 0$ and whole numbers.

The problem (1) – (4) is a generalization of comprehensive knapsack problem. In case if $n = N$ it ends up being absolutely comprehensive problem.

In this work a method of solving mixed-integer programming problems with one limit is offered. In this prospect, first, the number of variables and the range of possible values, which consists the solution, decreases. Further, this solution brings us to the problem which asks for the"branch and bounds" method of solution, where in each branch we limit the range of functional and the variables. Many held experiments show that the offered method works more efficiently than famous method of "branch and bounds".

Without interrupting commonness, we suppose that in the problem (1) – (4) the unknown variables are in the following order:

$$\frac{c_{j_{(1)}}}{a_{j_{(1)}}} \ge \frac{c_{j_{(2)}}}{a_{j_{(2)}}} \ge \dots \ge \frac{c_{j_{(k)}}}{a_{j_{(k)}}} \ge \dots \ge \frac{c_{j_{(N)}}}{a_{j_{(N)}}}$$

Then, the best solution $\overline{x} = (\overline{x_1}, \overline{x_2}, \dots, \overline{x_n})$ of the limitless problem (1) – (3) without variables, is analytically based on the next formula: for each i ($i = 1,2,\dots N$)

$$\overline{x}_{j_{(i)}} = \begin{cases} d_{j_{(i)}},\ if\ \ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \overline{x}_{j_{(r)}}) / a_{j_{(i)}} \ \ge\ d_{j_{(i)}}\ , \\ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \overline{x}_{j_{(r)}}) / a_{j_{(i)}},\ \text{if}\ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \overline{x}_{j_{(r)}}) / a_{j_{(i)}} \ <\ d_{j_{(i)}},\ k := i, \\ 0,\ \text{when}\ \ \ i = k+1, k+2, \dots, N. \end{cases}$$

We have to notice, what if $n < j_{(k)} \le N$ then the solutions $\overline{x} = (\overline{x_1}, \overline{x_2}, \dots, \overline{x_N})$ of boundless problem (1) – (3) are also the solutions of problem (1) – (4) and the process of the solving is over. Some approximate solution such as $\underline{x} = (\underline{x_1}, \underline{x_2}, \dots, \underline{x_N})$ of the problem (1) – (4) we can review for each i in the following way:

$$\underline{x}_{j_{(i)}} = \begin{cases} d_{j_{(i)}},\ if\ \ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \underline{x}_{j_{(r)}}) / a_{j_{(i)}} \ \ge\ d_{j_{(i)}}\ , \\ [(b - \sum_{r=1}^{i-1} a_{j_{(r)}} \underline{x}_{j_{(r)}}) / a_{j_{(i)}}],\ \text{if}\ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \underline{x}_{j_{(r)}}) / a_{j_{(i)}} \ <\ d_{j_{(i)}} \end{cases}$$

And for $i = n+1, n+2, ..., N$,

$$\underline{x}_{j_{(i)}} = \begin{cases} d_{j_{(i)}}, \ if \ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \underline{x}_{j_{(r)}}) / a_{j_{(i)}} \geq d_{j_{(i)}}, \\ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \underline{x}_{j_{(r)}}) / a_{j_{(i)}}, \ \text{if} \ (b - \sum_{r=1}^{i-1} a_{j_{(r)}} \underline{x}_{j_{(r)}}) / a_{j_{(i)}} < d_{j_{(i)}}, \ (k := i), \\ 0, \ \text{when} \quad i = k+1, k+2, ..., N. \end{cases}$$

So, for the most optimal solution of f^* for the task (1) – (4) we can determine the upper ($\overline{f}$) and the lower ($\underline{f}$) limits in the following way:

$$\overline{f} = \sum_{i=1}^{N} c_{j_{(i)}} \overline{x}_{j_{(i)}}, \quad \underline{f} = \sum_{i=1}^{N} c_{j_{(i)}} \underline{x}_{j_{(i)}}.$$ It is obvious that $\underline{f} \leq f^* \leq \overline{f}$.

<u>Theorem 1.</u> Let $h_{j_{(i)}} = (\overline{f} - \underline{f}) / \left| S_{j_{(i)}} \right|$, $i = \overline{1,N}$. Then, the coordinates of the optimal solution of the given problem (1) – (4) are located on the following interval:

a) If $i \in [1; k-1]$, then $x_{j_{(i)}} \in [\max\{0; d_{j_{(i)}} - [h_{j_{(i)}}]\}; d_{j_{(i)}}]$,

b) If $i \in [k+1; N]$, then $x_{j_{(i)}} \in [0; \min\{[h_{j_{(i)}}], d_{j_{(i)}}\}]$

Here $[z]$ - stands for the whole part of z, and $S_{j_{(i)}} = c_{j_{(i)}} - \dfrac{c_{j_{(k)}}}{a_{j_{(k)}}} a_{j_{(i)}}$, $i = \overline{1,N}$.

So, due to the use of the Theorem 1, the number of variables and the range of possible solutions, which includes the most optimal solution, decreases.

Let's consider that after using the Theorem 1 we end up having an equivalent problem such as:

$$\sum_{j=1}^{M} c_j y_j \to \max \quad, \tag{5}$$

$$\sum_{j=1}^{M} a_j y_j \leq b', \tag{6}$$

$$a_j \leq y_j \leq d'_j, \quad j = \overline{1,M}, \tag{7}$$

$$y_j, d'_j - \text{the whole numbers } j = \overline{1,m}. \tag{8}$$

Here $M \leq N, \ m \leq n, \ b' \leq b \ u \ \ y_j = x_j - \alpha_j \quad d'_j = \beta_j - \alpha_j, \quad j = \overline{1,M}$

The values α_j and β_j $j = \overline{1,M}$ can be determined by the Theorem 1.

The experiments show that the number of mixed-integer variables m in the problem (5) – (8) are a lot less than the number of n in the original problem (1) – (4).

Let's consider $\overline{Y} = \left(\overline{y}_1, \overline{y}_2, \cdots \overline{y}_M \right)$ – to be the optimal solution of limitless problem (5) – (8), which is determined analytically below, $\underline{g}$ and $\overline{g}$ – are the lower and the upper bounds of optimum in this problem. Generally we can consider that the division $c_{j(i)} / a_{j(i)}$ variables are not in the order of increasing.

When the most optimal solution $\overline{Y}$ of the unlimited problem (5) – (8), is being solved the following way: for each i $(i=\overline{1,M})$

$$\overline{y}_{j(i)}=\begin{cases} d'_{j(i)} \; if \quad and \;_{j(i)} \; d'_{j(i)} \le b' - \sum_{r=1}^{i-1} a_{j(i)}\overline{y}_{j(i)}, \\ \left(b' - \sum_{r=1}^{i-1} a_{j(r)}\overline{y}_{j(r)}\right) \Big/ a_{j(i)}, \; ifand \;_{j(i)} \; d'_{j(i)} > b' - \sum_{r=1}^{i-1} a_{j(r)}\overline{y}_{j(r)} \\ 0, \quad ïðè \quad i=k+1, k+2, \cdots, M \end{cases}$$

Here k is a variable, being represented as ratio.

Also, we have to notice, that if $m < j(k) \le M$, then the solution $\overline{Y}=(\overline{y}_1, \overline{y}_2, \cdots \overline{y}_M)$ of the unlimited problem (5) – (7) is also the most suitable solution of the mixed-integer problem (5) – (8)and the process of solving is over . That is why we suppose that $j(k) \le m$.

Considering all the conventional signs from the Theorem 1we can draw a conclusion.

Statement 1. Coordinates $y_{j(i)}$, $i=1,M$ the optimal solutions of the problem (5) – (8) are in the interval:

$$Y_{j(i)} \in \left[\max \left\{(0; d'_{j(i)}) - \left[h'_{j(i)}\right]\right\}); \; d'_{j(i)})\right], \; when \;\; 1 \le i \le k-1,$$
$$Y_{j(i)} \in \left[0; \min \left\{\left[h'_{j(i)}\right]); \; d'_{j(i)})\right\}\right], \; when \;\; k+1 \le i \le m.$$

Here $[h'_{j(i)}]$ stands for the whole number $h'_{j(i)}$,

$$h'_{j(i)} = \left(\overline{g} - \underline{g}\right) / \left| S_{j(i)} \right|, \quad S_{j(i)} = c_{j(i)} - \frac{c_{j(k)}}{a_{j(k)}} \cdot a_{j(i)}, \; i=\overline{1,M}.$$

From the Statement 1 we have, that if for some $i_* < k$ the lower bound $y_{j(i_*)}$ is $d'_{j(i_*)}$, then it is the most optimal solution $y_{j(i)} = d'_{\;j(i)}$ for all $i=i_*$, $i=i_*-1$, $i=i_*-2, \cdots, i=1$. And if $i_* > k$ the upper bound $y_{j(i_*)}$ will be zero, then the most optimal solution of the mixed-integer problem (5) – (8) $y_{j(i)}=0$, when $i=i_*$, $i=i_*+1, \cdots, M$.

Let's note, that during numeral experiments this type of situations occur often so the number of branches considerably decreases.

<u>Theorem 2.</u> a) If there is natural g_2, which corresponds to condition $\overline{g} - S_{j(k-1)} < g_2 < \overline{g}$, then in the solution (5) – (8) $y_{j(i)} = d'_{j(i)}$, when $i=1,2,...,k-1$.

б) If there is natural g_2, which corresponds to condition $\overline{g} - \left|S_{j(k+1)}\right| < g_2 < \bar{f}$, then in the solution (5) – (8) $y_{j(i)}=0$, when $i=\overline{k+1,M}$.

Here k -is the number of the variable which is a ratio in the solution of mixed-integer problem (5) – (7).

<u>Theorem 3.</u> If for some number g_2, the conditions of the Theorem 2 can be applied and $[\widetilde{y}_{j(k)}] < \overline{y}_{j(k)}$, then g_2 will be lowered upper limit of the functional of the problem (5) – (8), therefore $g_2 < [\overline{g}]$. Here

$$\widetilde{y}_{j(k)} = (b' - \sum_{i=1}^{k-1} a_{j(i)} \cdot d'_{j(i)}(g_2)) \Big/ a_{j(k)}, \qquad d'_{j(i)}(g_2) = d'_{j(i)} - \left[\frac{\overline{g} - g_2}{S_{j(i)}}\right], \qquad i=\overline{1,k-1}.$$

It is necessary to note that the process of finding bettered upper limit g_2 is essential for the method of "branch and bounds". From the other side, if there is no natural number such as

g_2, which corresponds to the conditions of the Theorem 2 then the upper limit $\overline{g}$ does not decrease.

Statement 2. If for some natural g_2, where $g_2 = \underline{g}$ corresponds to the conditions from the Theorem 3, then the solution $y_{j(i)} = d'_{j(k)}$, $i = \overline{1, k-1}$, $y_{j(k)} = \left[\bar{y}_{j(k)}\right]$,

$$y_{j(k+1)} = (b' - \sum_{r=1}^{k} a_{j(r)} \cdot y_{j(r)}) \Big/ a_{j(k+1)}, \quad y_{j(i)} = 0, \ i = \overline{k+2, M},$$

is the most optimal solution of the mixed- integer problem (5) – (8).

The most distinguishing features of this method are: first we use the Theorem 1 to create the equaling problem (5) – (8) which has les variables and more limited range. Further we check the conditions of Theorem 3 and Statement 2. If the conditions of the Statement 2 suit then the optimal solution is found. If not, we have to build additional problems a) and б) according to k which is a ratio.

$$y_{j(k)} \le [\bar{y}_{j(k)}], \qquad y_{j(k)} \ge [\bar{y}_{j(k)}] + 1$$

Each of the following problems a) and б) are solved analytically without the condition of whole parts of variables. Also bettered upper limits are found.

It is important to notice, that for each unsolved problem from the list the range of variables and limits decreases. That is why, based on this conclusion we can say that the method offered in this paper is more efficient that classical method of "branch and bounds.

Notice, that the results we get in this paper are generalization of the work [1 – 5] to more range of problems particularly for mixed-integer programming with only one limit.

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