The Third International Conference "Problems of Cybernetics and Informatics", September 6-8, 2010, Baku, Azerbaijan. Section #7 "Numerical Methods and Computational Technology"
www.pci2010.science.az/7/04.pdf

ON THE NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL EQUATION

Galina Mehdiyeva[1], Vagif Ibrahimov[2], Mehriban Imanova[3]

Baku State University, Baku, Azerbaijan
[1,2]ibvag@yahoo.com, [3]imn_bsu@mail.ru

Considering wide application of integro-differential equation, last time scientists from different countries have actively engaged in research of approximate solutions of integro-differential equations. Many of these scientists mainly are engaged in numerical solution of Volterra integro-differential equations. For this purpose, quadratic method, modification of the quadratic method, collocation method, etc. were investigated. Here we consider the use of implicit multistep methods with constant coefficients for solving Volterra integro-differential equations, and find sufficient conditions for its convergence.

Introduction. Consider the following initial-value problem for the integro-differential equations of Volterra type:

$$y'(x) = f(x,y) + \int_{x_0}^{x} K(x,s,y(s))ds, \ \ y(x_0) = y_0. \tag{1}$$

Let's suppose continuous functions $f(x,y)$ and $K(x,s,y)$ are defined on some closed domain G and the problem (1) has the unique solution defined on the segment $[x_0, X]$. It is known that the problem (1) can be replaced by the next problem:

$$y'(x) = f(x,y) + v(x), \quad y(x_0) = y_0, \tag{2}$$

$$v(x) = \int_{x_0}^{x} K(x,s,y(s))ds. \tag{3}$$

For solving the system which consists of the relation (2) and (3), here we suggested following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\sum_{i=0}^{k} \beta_i v_{n+i}, \tag{4}$$

$$\sum_{i=0}^{k} \alpha'_i v_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k} \gamma_{i,j} K(x_{n+j}, x_{n+i}, y_{n+i}), \tag{5}$$

where the coefficients $\alpha_i, \alpha'_i, \beta_i, \gamma_{i,j}$ $(i,j = 0,1,....,k)$ are some real numbers, and $\alpha_k \neq 0, \alpha'_k \neq 0$, a constant 0 < h is a step of partitioning of the segment $[x_0, X]$ into N equal parts, and the mesh points are defined in the form $x_i = x_0 + ih$ $(i = 0,1,...,N)$.

Here, at the mesh points $x_m (m = 0,1,2,...)$, the approximate values of the functions $y(x)$ and $v(x)$ are denoted by y_m and v_m $(m = 0,1,2,...)$, but exact values by $y(x_m)$ and $v(x_m)$ $(m = 0,1,2,...)$, respectively.

One of the important questions at researching method (4)-(5) is how much are closer the found values of the solution of a problem (1) by scheme (4) - (5) to exact values of the solution. Therefore consider convergence of the method (4) — (5) to the solution of the problem (1).

§1. Convergence of k-step method with the constant coefficients.

As is known, at real use of k -step method defined by scheme (4) and (5) there arises some round-off errors. Hence, the k -step method at real use will have the following form (see, f.e. [1]-[3]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\beta_i v_{n+i} + \delta_n, \tag{1.1}$$

$$\sum_{i=0}^{k}\alpha_i' v_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_{i,j}K(x_{n+j}, x_{n+i}, y_{n+i}) + \delta_n'. \tag{1.2}$$

Here the round-off errors for the corresponding method are designated by δ_n and δ_n'.

As the sizes y_m and v_m $(m = 0,1,2,...)$ are the approximate values of the solution of problem (2) and equation (3), it is natural that at replacement by their exact values in (4) and (5), we will receive some error which usually is called an error of methods. We have:

$$\sum_{i=0}^{k}\alpha_i y(x_{n+i}) = h\sum_{i=0}^{k}\beta_i f(x_{n+i}, y(x_{n+i})) + h\sum_{i=0}^{k}\beta_i v(x_{n+i}) + R_n, \tag{1.3}$$

$$\sum_{i=0}^{k}\alpha_i' v(x_{n+i}) = h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_{i,j}K(x_{n+j}, x_{n+i}, y(x_{n+i})) + R_n'. \tag{1.4}$$

R_n and R_n' are the errors of corresponding methods. Somebody can find the coefficients of method (5) by the way from [4].

The convergence of method (4)-(5) can be defined by means of the following theorem.

Theorem 1. Let the following conditions be satisfied:

1. Methods (4) and (5) are stable, and $\alpha_k \neq 0$, $\alpha_k' \neq 0$.
2. Methods (4) and (5) have degree p.
3. Continuity on totality of variables of the functions $f(x, y)$ and $K(x, s, y)$ are defined in some closed domain they have partial derivatives to some p, inclusively.
4. The smallness order for the initial data y_m $(m = 0,1,...,k-1)$ has the following form

$$y_m - y(x_m) = O(h^p),\ \ v_m - v(x_m) = O(h^p).$$

5. Round-off errors have higher order smallness, than the initial data, namely:

$$\delta_n = O(h^{p+1}),\quad \delta_n' = O(h^{p+1}),\ \ h \to 0.$$

Then it holds

$$\max_{v}((y(x_v) - y_v), (v(x_v) - v_v)) = O(h^p),\ \ h \to 0. \tag{1.5}$$

Here the concept of stability and degree of a multistep method are defined just as in [5].

Proof. Subtracting (1.1) and (1.2) from (1.3) and (1.4), we have:

$$\varepsilon_{n+k} = \sum_{i=0}^{k-1}\frac{-\alpha_i + a_i h + b_i h^2}{\alpha_k - a_k h - b_k h^2}\varepsilon_{n+i} + h\sum_{i=0}^{k-1}\bar{\beta}_i\bar{\varepsilon}_{n+i} + \hat{R}_n, \tag{1.6}$$

$$\bar{\varepsilon}_{n+k} = -\sum_{i=0}^{k-1}\frac{\alpha_i'}{\alpha_k'}\bar{\varepsilon}_{n+i} + \frac{h}{\alpha_k'}\sum_{i=0}^{k}b_i\varepsilon_{n+i} + \hat{R}_n', \tag{1.7}$$

where

$$a_i = \beta_i f_y'(x_{n+i}, \xi_{n+i}),\ \ b_i = (\beta_k/\alpha_k')\sum_{j=0}^{k}\gamma_{i,j}K_y'(x_{n+j}, x_{n+i}, \eta_{n+i}),$$

$$\hat{R}_n' = R_n' - \delta_n';$$

$$\hat{R}_n = h\beta_k\hat{R}_n'/\alpha_k' + R_n - \delta_n/(\alpha_k - a_k h - b_k h^2),$$

$$\varepsilon_m = y(x_m) - y_m,\ \ \bar{\varepsilon}_m = v(x_m) - v_m,\ \ \bar{\beta}_i = \frac{\beta_i - \beta_k\alpha_i'/\alpha_k'}{\alpha_k - a_k h - b_k h^2},$$

ξ_{n+i} and η_{n+i} are between y_{n+i} and $y(x_{n+i})$.

After some transformations, we can write relation (1.6) and (1.7) in the following form:

$$\varepsilon_{n+k} = -\sum_{i=0}^{k-1} \frac{\alpha_i}{\alpha_k} \varepsilon_{n+i} + h\sum_{i=0}^{k-1} \bar{\beta}_i \bar{\varepsilon}_{n+i} + h\sum_{i=0}^{k-1} e_i \varepsilon_{n+i} + h^2 \sum_{i=0}^{k-1} l_i \varepsilon_{n+i} + \hat{R}_n, \tag{1.8}$$

$$\bar{\varepsilon}_{n+k} = -\sum_{i=0}^{k-1} \frac{\alpha_i'}{\alpha_k'} \bar{\varepsilon}_{n+i} + h\sum_{i=0}^{k-1} (b_i + b_k d_i)\varepsilon_{n+i} + h^2 b_k \sum_{i=0}^{k-1} \bar{\beta}_i \bar{\varepsilon}_{n+i} + \hat{R}_n' + \frac{hb_k}{\alpha_k'} \hat{R}_n, \tag{1.9}$$

where

$$e_i = \frac{-\alpha_i a_k + a_i \alpha_k}{\alpha_k(\alpha_k - a_k h - b_k h^2)},\ \ l_i = \frac{-\alpha_i b_k + b_i \alpha_k}{\alpha_k(\alpha_k - a_k h - b_k h^2)},\ d_i = -\frac{\alpha_i}{\alpha_k} + h(e_i + hl_i).$$

After addition of the next identities:

$$\varepsilon_{n+i} = \varepsilon_{n+i},\quad \bar{\varepsilon}_{n+i} = \bar{\varepsilon}_{n+i}\ (i = 0,1,2,...,k-1)$$

rewrite the received system of the equations by means of the following vectors

$$Y_{n+k} = (\varepsilon_{n+k}, \varepsilon_{n+k-1}, ...\varepsilon_{n+1}),\quad \bar{Y}_{n+k} = (\bar{\varepsilon}_{n+k}, \bar{\varepsilon}_{n+k-1}, ...\bar{\varepsilon}_{n+1})$$

in the form:

$$Y_{n+k} = AY_{n+k-1} + hB\bar{Y}_{n+k-1} + hB_1 Y_{n+k-1} + h^2 B_2 Y_{n+k-1} + W_n, \tag{1.10}$$

$$\bar{Y}_{n+k} = A'\bar{Y}_{n+k-1} + hVY_{n+k-1} + h^2 V_1 \bar{Y}_{n+k-1} + \bar{W}_n, \tag{1.11}$$

where

$$A = \begin{pmatrix} -\frac{\alpha_{k-1}}{\alpha_k} & -\frac{\alpha_{k-1}}{\alpha_k} & \dots & -\frac{\alpha_1}{\alpha_k} & -\frac{\alpha_0}{\alpha_k} \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},\ A' = \begin{pmatrix} -\frac{\alpha_{k-1}'}{\alpha_k'} & -\frac{\alpha_{k-1}'}{\alpha_k'} & \dots & -\frac{\alpha_1'}{\alpha_k'} & -\frac{\alpha_0'}{\alpha_k'} \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} \bar{\beta}_{k-1} & \bar{\beta}_{k-1} & \dots & \bar{\beta}_0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} e_{k-1} & e_{k-2} & \dots & e_0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} l_{k-1} & l_{k-2} & \dots & l_0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} v_{k-1} & v_{k-2} & \dots & v_0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, V_1 = \begin{pmatrix} \bar{\beta}_{k-1} & \bar{\beta}_{k-2} & \dots & \bar{\beta}_0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix},\quad \bar{W}_{n,k} = \begin{pmatrix} \bar{R}_n \\ 0 \\ \vdots \\ 0 \end{pmatrix};\quad W_{n,k} = \begin{pmatrix} \breve{R}_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$v_i = (b_i + b_k \bar{d}_i)/\alpha_k,\quad \bar{d}_i = (-\alpha_i + a_i h + b_i h^2)/(\alpha_k - a_k h - b_k h^2).$$

By the vector $Z_{n+k} = (Y_{n+k}, \bar{Y}_{n+k})$ we will write the equations (1.10) and (1.11) in the following form:

$$Z_{n+k} = \bar{A} Z_{n+k-1} + h\bar{B} Z_{n+k-1} + h^2 \breve{B} Z_{n+k-1} + \breve{W}_n, \tag{1.12}$$

where

$$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix},\qquad \bar{B} = \begin{pmatrix} B_1 & B \\ V & 0 \end{pmatrix},\qquad \breve{B} = \begin{pmatrix} B_2 & 0 \\ 0 & V_1 \end{pmatrix},\qquad \breve{W}_n = \begin{pmatrix} W_n \\ \bar{W}_n \end{pmatrix}.$$

On theorem conditions, characteristic numbers of a matrix A and A' are in a unit circle on whose border there are no multiple roots. Then there is a block matrix C such that $\|D\| = \|C^{-1}\bar{A}C\| \le 1$ [6].

In the equation (1.12) we use replacement $Z_m = CT_m$. Then, multiplying the received equation at the left by C^{-1}, we have:

$$T_m = DT_{m-1} + hS_1T_{m-1} + h^2 S_2 T_{m-1} + w_m. \qquad (1.13)$$

Here,

$$D = C^{-1}\overline{A}C,\ S_1 = C^{-1}\overline{B}C,\ S_2 = C^{-1}\breve{B}C,\ w_m = C^{-1}\breve{W}_n.$$

It is easy to show that the norm of matrices S_1, S_2 and a vector w_m are limited i.e.

$$\|S_1\| \le M_1,\ \|S_2\| \le M_2,\ \|w_m\| \le M_3 \max(R_m, R'_m)\ .$$

Passing in norm in the equation (1.13), we have

$$\|\mathrm{T}_m\| \le M_3 \left|\overline{R}_j\right| + (1+h\gamma)\|T_{m-1}\|, \qquad (1.14)$$

where $\overline{R}_m = \max(R_m, R'_m),\ \ \gamma = M_1 + hM_2$.

By means of mathematical induction, from (1.14) we will receive

$$\|\mathrm{T}_m\| \le M_3 \sum_{j=k}^{m} (1+h\gamma)^{m-j} \left|\overline{R}_j\right| + (1+h\gamma)^{m-k+1} \|\mathrm{T}_{k-1}\|. \qquad (1.15)$$

Simplifying this estimation and using the following inequality

$$(1+h\gamma)^{m-j} \le \exp(h\gamma(m-j)) \le \exp(h\gamma m)$$

from (1.15) we have:

$$\|T_m\| \le \exp(h\gamma m)(M_3 \sum_{j=k}^{m} \left|\overline{R}_j\right| + \|T_{k-1}\|).$$

It is clear that the next inequality is valid:

$$\max(\varepsilon_m, \overline{\varepsilon}_m) \le \|Z_m\| \le \|C\| \|T_m\|.$$

Hence

$$\max(\varepsilon_m, \overline{\varepsilon}_m) \le \exp(h\gamma m)(M_3 \|C\| \sum_{j=k}^{m} \left|\overline{R}_j\right| + \|C\| \|T_{k-1}\|).$$

From here

$$\max(\varepsilon_m, \overline{\varepsilon}_m) \le \exp(\gamma X m)(\overline{M}_3 \sum_{j=k}^{m} \left|\overline{R}_j\right| + \|C\| \|C^{-1}\| \max_{0 \le i \le k-1} (\varepsilon_i, \overline{\varepsilon}_i)). \qquad (1.16)$$

Considering in (1.16) theorem conditions, we will receive

$$\max(\varepsilon_m, \overline{\varepsilon}_m) = O(h^p), \quad h \to 0.$$

The theorem is proved.

References

1. M.N. Imanova. Numerical solution of Volterra integro-differential equation. Proceeding of institute of mathematics and mechanics, XXII, Baku, 2005.
2. E. Hairer, S.P.Norsett, G.Wanner. Solving ordinary differential equations. (In Russian) M., Mir, 1990, 512 p.
3. A.F. Verlan, V.S. Sizikov. Integral equations: methods, algorithms, programs. (In Russian) Naukova Dumka, Kiev (1986), 544 p.
4. G.Yu.Mehdiyeva, V.R.Ibrahimov. On a way of the definition of the coefficients multistep methods. (In Russian). News of Baku University, Series of physico-math.sciences, 2008, №2, pp. 35-39.
5. G. Dahlquist. Convergence and stability in the numerical integration of ordinary differential equations. Math. Scand. - 1956, №4, pp. 33-53.
6. N.S Bakhvalov. Numerical methods. (In Russian) Nauka, 1973, 631 p.