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ON THE DECREASING OF INFORMATION MEASURES IN THE INFORMATION PRECISION PROCESS

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A general model for decision problems is presented by a basic probability assignment of a body of evidence, which gives the information on distribution of states, situations or factors in the form of Dempster-Shafer belief structure. The rule for decision making is constructed from two steps by means of a composition of two functions –Dempster's lower and upper expected values.

On the decision-making system. Some decision problems can be considered as given by the decision-making information system:

$$(\Omega, D, I, u, K)$$

where Ω is the non-empty set of the states (acts, factors, situations, symptoms and so on) of nature; D is non-empty set of the feasible decisions (possible alternatives); I is the available information about Ω; K is the decision-maker's criterion, which represents some optimal principle; and $u: D\times\Omega \to R$, is a valuation of the consequences, coherent with the decision-maker's preferences (utilities, results, gains and so on).

According to the kind and amount of available information I, the following cases have been distinguished:

- General Decision Problem in a Certain Environment: when the state of nature which will occur is known "a priori".
- General Decision Problem in a Risk Environment: if the true state is unknown but a probability distribution is available on Ω.
- General Decision Problem in an Uncertain Environment: when no information about the states of nature can be used.

Our aim in this work is to study a more general model including the previous three, such a model will consider the information about Ω as defined by a body of evidence ([2,3,4,5] and so on).

To obtain a solution for a decision problem as defined above, an order relation should be found on the set of decisions D; we will construct this order taking into account the decision-maker's preference valuations u and the information I. We suppose D and Ω to be finite, in order to avoid measurability or convergence problems. If we denote

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}, \quad D = \{d_1, d_2, ..., d_m\}, \tag{1}$$

the consequences of a decision d_i are given in terms of a utility vector $\bar{u}_i$:

$$d_i \leftrightarrow \bar{u}_i = (u_{i1}, u_{i2}, ..., u_{in}) \in R^n, \ (i = 1, 2, ..., m), \tag{2}$$

which represents the decision-maker's preferences.

The problem is now to find an order on R^n. Classically the solution is obtained by mapping each vector $\bar{u}_i$ on a value of R; to build this map $\varphi: R^n \to R$, we will use the decision-maker's opinions and the information available about Ω.

Thus, we will say that a decision d_i is preferred or indifferent to another d_k (and express it as):

$$d_k \preccurlyeq d_i \Leftrightarrow (u_{k1}, u_{k2}, ..., u_{kn}) \le (u_{j1}, u_{j2}, ..., u_{jn}) \Leftrightarrow \quad \varphi(\bar{u}_k) \le \varphi(\bar{u}_i). \tag{3}$$

Numerous examples of this procedure exist in the relevant literature, as the criteria K of the expected value (risk environment), Laplace, Wald (uncertain environment), etc.

Some basic definitions. According to Dempster-Shafer Theory of Evidence we will suppose the information I is given by a body of evidence represented by a basic probability assignment (B.P.A.) [2,3]

Definition 1. a) A B.P.A. On Ω is a map $m: 2^{\Omega} \to [0;1]$,
fulfilling the conditions:

$$\text{(i)} \quad m(\varnothing) = 0,$$
$$\text{(ii)} \quad \sum_{A \subset \Omega} m(A) = 1. \qquad (4)$$

b) Every $A \in 2^{\Omega}$ for which $m(A) > 0$ is usually called a focal element of m. If $\Im$ denotes the set of all focal elements, then the pair $< \Im, m >$ is called a Body of Evidence.

Definition 2. Let m be a B.P.A. on Ω. The plausibility Pl and belief Bel measures associated to m are given by the formula

$$Pl(A) = \sum_{A \cap B \neq \varnothing} m(B), \quad Bel(A) = \sum_{B \subset A} m(B), \ \forall A \in 2^{\Omega}. \qquad (5)$$

The relationship between $m(A)$ and $Bel(A)$ has the following meaning – whereas $m(A)$ characterizes the degree of evidence or belief that the element in question belongs to the set A alone (i.e. exactly to set A), $Bel(A)$ represents the total evidence of belief that the element belongs to A as well as to the various special subsets of A. The plausibility measures $Pl(A)$ has a different meaning: it represents not only the total evidence or belief that the element in question belongs to set A or to any of its subsets, but also the additional evidence or belief associated with sets that overlap with A. Hence $Pl(A) \geq Bel(A)$.

In 1967, Dempster [1] introduced the concepts of lower and upper expected values of a function, with respect to a measure as a generalization of the expected mathematical value:

Definition 3. let $h: \Omega \to R$ be any function and let m be a B.P.A. on Ω. Lower and upper expected values of h with respect to m are defined as

$$E_*(h/m) = \sum_{A \subset \Omega} m(A) \cdot \inf_{\omega \in A} h(\omega), \qquad E^*(h/m) = \sum_{A \subset \Omega} m(A) \cdot \sup_{\omega \in A} h(\omega). \qquad (6)$$

Construction of information inclusion relation. Let us consider two sets of information about it, each one being represented by a B.P.A.'s - m_1 and m_2. The analysis of possible relations existing between them constitutes an immediate problem, the most natural relation is inclusion (or relation of more precise information) - $\subset$. We will define a set of information, m_1, as included in another set, m_2, if knowledge provided by m_1 about the unknown element of the set Ω is less precise than that given by m_2.

Definition 4. If m_1 and m_2 are defined on Ω, the evidence associated to m_1 is included in the evidence associated to m_2 ($m_1 \subset m_2$) if $\forall A \subset \Omega, \exists B.P.A. - m_A : 2^{\Omega} \to [0,1]$, verifying

$$m_1(A) = \sum_{B: B \subset A} m_A(B) \text{ and } m_2(B) = \sum_{A: B \subset A} m_A(B). \qquad (7)$$

This definition is based on an intuitive idea according to which any additional available information must be result in an atomization (m_2) of the earlier evidence (m_1). Some particular cases of Evidence give rise to interesting situation which are to be found in the following propositions:

Proposition 1. Let be given the evidence of total ignorance, corresponding B.P.A. by m_0

$$m_0(A) = \begin{cases} 1, & \text{if } A = \Omega, \\ 0, & \text{otherwise.} \end{cases} \qquad (8)$$

Let m_0 be the probabilistic evidence and let m be any B.P.A. such that $m_0 \subset m \subset m_p$, then for any decision d_i we have

1. $E_*(\bar{u}_i/m_0) = \min_j(\bar{u}_i)$, $E^*(\bar{u}_i/m_0) = \max_j(\bar{u}_i)$,

2. $E_*(\bar{u}_i/m_p) = E^*(\bar{u}_i/m_p) = E(\bar{u}_i)$,

3. $[E_*(\bar{u}_i/m_p); E^*(\bar{u}_i/m_p)] \subset [E_*(\bar{u}_i/m); E^*(\bar{u}_i/m)] \subset [E_*(\bar{u}_i/m_0); E^*(\bar{u}_i/m_0)]$ (interval inclusion),

where E is a symbol of a mathematical expectation.

Proposition 2. Let $m_1 \subset m_2$ be two B.P.A. on Ω. The interval inclusion relation

$$[E_*(\bar{u}_i/m_2); E^*(\bar{u}_i/m_2)] \subset [E_*(\bar{u}_i/m_1); E^*(\bar{u}_i/m_1)] \quad (9)$$

is verified for any fixed decision – d_i.

Proposition 3. If m_1 and m_2 are two B.P.A. on Ω such that $m_1 \subset m_2$, then

$$[Bel_2(A);\ Pl_2(A)] \subset [Bel_1(A);\ Pl_1(A)],\ \forall A \in 2^{\Omega}. \quad (10)$$

Decision-Making scaling criteria based on the Dempster's extremal expectations. In the conditions presented here, we can map the vector of valuations of decision-maker's preferences $\bar{u}_i \in R^n$ on another vector of R^2 by means of $t: R^n \to R^2$,

$$t(u_{i1}u_{i2},...,u_{in}) = (E_*(\bar{u}_i/m), E^*(\bar{u}_i/m)). \quad (11)$$

If we consider the composition $h \circ t = \varphi$, the determination of φ means merely to determine the map: $h: R^2 \to R$. From this composition, one can see t contains the available information while h must reflect the decision maker's attitude.

Finally, we may note: On one hand, if a body of evidence considered about Ω is probabilistic ($m \equiv m_p$), then

$$E_*(\bar{u}_i / m_p) = E^*(\bar{u}_i / m_p) = E_{m_p}(\bar{u}_i),$$

where the most outstanding ways to define h are as follows:

(a) Optimistic criterion based on the map h^*: $h = h^*(E_*, E^*) = E^*$

(b) Pessimistic criterion based on the map h_*: $h = h_*(E_*, E^*) = E_*$

On the other hand, if we confront a problem in the absence of information, the only possible body of evidences to be considered is the so called total ignorance $(m = m_0)$, and in this case

$$E_i^* = \max_{\omega_j = \Omega} \bar{u}_i = \max_j u_{ij}, \quad E_{i*} = \min_{\omega_j = \Omega} \bar{u}_i = \min_j u_{ij}$$

are verified. If the decision maximizing h is chosen (criterion K), we find:

(a) The max-max criterion, from h^*:

$$\max_i E_i^* = \max_i E^*(\bar{u}_i / m_0) = \max_i \max_j u_{ij}$$

as a particular case of our optimistic criterion.

(b) Wald's criterion, or max-min criterion, from h_*:

$$\max_i E_{i*} = \max_i E_*(\bar{u}_i / m_0) = \max_i \min_j u_{ij}$$

as a particular case of our pessimistic criterion.

Conclusion. Dempster-Shafer's mathematical theory of a body of evidence is a powerful tool to build modeling decisions in risk or uncertain environments. By expressing the available information about states or factors of nature in a decision problem by means of a body of evidence and by using the lower and upper expected values to obtain decision rules, one may generalize classical criteria to intermediate situations between null and probabilistic sets of information.

A definition of more precise information (inclusion) relation on the set of evidences is used to study existing relations among results which can be obtained from different B.P.A.

The case of total ignorance and probabilistic evidence are extreme in the sense the first one produces maximum difference between both terms of Dempster's extremal expectation for each decision and the second one reduces the range to zero.

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