ON EXISTENCE OF SADDLE POINT OF LAGRANGE FUNCTION FOR A MATHEMATICAL PROGRAMMING PROBLEM IN BANACH SPACE

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There are many work devoted to mathematical programming in infinite dimensional spaces [1, 3, 4, 5, 6, 7]. In most of them the proof of an existence of saddle point of Lagrange function is required existence of nonempty interior of the cone which defines partial order in the space. Then under some conditions (Slater, regularity, etc.) existence of saddle point of Lagrange function or Kuhn-Tucker conditions are established. Unfortunately, natural cones of many important spaces, such as $L_p[0, T]$ and $l_p (1 < p < \infty)$ have no interior points.

In this paper we prove an existence of saddle point of Lagrange function for convex programming problem in Banach spaces.

Initial problem

Let *X* and *Y* be reflexive Banach spaces partial ordered by convex closed cones *K* and *P*, respectively. A is a linear bounded operator, mapping *X* to *Y*, *J(x)* – a continuously differentiable convex functional.
The problem is to minimize the functional subject to constraints

$$Ax \le b \quad (b - Ax \in P)$$
$$x \ge 0 \; (x \in K)$$

We will write this problem short, as

$$J(x) \to \min \quad (1)$$
$$\left.\begin{array}{l} Ax \le b \\ x \ge 0 \end{array}\right. \quad (2)$$

Definition 1. If there exists $\varepsilon_0 > 0$ such that for every $\bar{b} \in \{\| \bar{b} - b \| \le \varepsilon_0\}$ the system $Ax \le \bar{b}$, $x \ge 0$ has a solution, then we say that constraints (2) satisfy strong simultaneity condition.

Existence of a saddle point

To prove existence of saddle point, we first prove the following lemmas.
Lemma1. If constraints (2) are strong simultaneous (i.e. satisfy strong simultaneity condition), then the set

$$M = \{z \in Y : b - Ax \ge z, \; x \ge 0\}$$

have internal points. (Not that a point $p \in M$ is called an internal point of M, if for each $z \in Y$ there exists a real number $\varepsilon > 0$, such that $p + \lambda z \in M$ for $|\lambda| \le \varepsilon$).

Lemma 2. If constraints (2) are strong simultaneous, then the set

$$S = \{(z, \rho) \in Y \times R : b - Ax \ge z, \; J(x) \le \rho, \; x \ge 0\}$$

have internal points.
Let X^* and Y^* be the conjugate spaces of *X* and *Y*, respectively.
Conjugate cone of K is K^*

$$K^* = \{x^* \in X^*; (x^*, x) \ge 0 \text{ for each } x \in K\}$$

and conjugate cone of P is defined as well. Let X^* and Y are partial ordered by K^* and P^*, respectively.

Lemma 3. If constraints (2) are strong simultaneous, then for any $z^* \in P^*$, $z^* \neq 0$ there exists a point $x_{z^*} \geq 0$ such that

$$(z^*, b - Ax_{z^*}) > 0$$

Definition 2. A pair $\langle x_0, z_0^* \rangle$ is said to be saddle point of Lagrange function if $x_0 \geq 0,\ z_0^* \geq 0$ and

$$L(x_0, z^*) \leq L(x_0, z_0^*) \leq L(x, z_0^*) \text{ for each } x_0 \geq 0,\ z^* \geq 0. \qquad (6)$$

It is easy to show that existence of a saddle point of Lagrange function follows existence of a solution of problem (1), (2). Converse of this is not so trivial. So prove the following theorem.

Theorem 1. Suppose that constraints (2) are strong simultaneous, and the problem (1), (2) have a solution x_0. Then there exists a non-zero linear functional, z_0^*, such that the pair $\langle x_0, z_0^* \rangle$ is a saddle point of Lagrange function.

Note 1. It is easy to see that strong simultaneity condition of (2) is equivalent to following relation

$$0 \in \operatorname{int}(AK + b + P)$$

Clearly **AK + b + P** can have interior points even if P have no interior points. It means that strong simultaneity condition can take place also in cases when we cannot talk about Slater condition.

Note 2. If int P $\neq \varnothing$ Slater condition and strong simultaneity are equivalent.

References

1. Hurwitz L, Udzava H. On saddle point of Lagrange function. – Studies in investigation on linear and nonlinear programming – Stanford university press (1958).
2. Danford N, Schwartz S. T. Linear operations, Part 1: General theory. 1988.
3. Regina Sandra Burachik and Vaithilingam Jeyakumar A new geometric condition for Fenchel's duality in infinite dimensional spaces. – Mathematical Programming, Springer, V 104, № 2-3, 2005.
4. L. C. Evans and D. Gomes – Control, Optimization and Calculus of Variations. 2002, V 8.
5. K. Malanowski. Second – order conditions and constraint qualifications in stability and sensitivity analysis of solutions to optimization problems in Hilbert spaces. – J. Applied Mathematics and Optimization. V 25, № 2, 1992.
6. Shen-Yu Chen, Soon-Yi x/4. Algorithms for infinite quadratic programming in spaces. – Journal of Computational and Applied Mathematics. V 213, issue 2, 2008.
7. Satoshi Ito. A numerical approach to infinite dimensional linear programming in, spaces. – Journal of industrial and Management Optimization (JIMO). V 6, issue 1, 2010V.I. Yelisyeyev. Introduction into methods of the theory of functions of spatial complex variable. (In Russian) NIA Publishing House, Moscow (1990) 189 p. www.maths.ru
8. I.L. Kantor, A.S. Solodovnicov. Hypercomplex numbers. (In Russian) "Nauka" Publishing House, Moscow (1973) 144 p.
9. Hypercomplex numbers (In Russian), BSE, v. 6, Moscow (1971) 536 p.
10. E. Madelung. The mathematical apparatus of physics. (In Russian) The State Publishing House of physical and mathematical literature, Moscow (1960) 620 p.
11. L.S. Pontryagin. Generalization of numbers. (In Russian) Moscow, "Nauka" ("Kvant", issue 54) (1986) 120 p.