SOLUTION TO AN OPTIMAL CONTROL PROBLEM WITH RESPECT TO QUASI-LINEAR HEAT CONDUCTION EQUATION

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The study and determination of the optimal regimes of heat conduction processes at a long interval of the change of temperature gives rise to optimal control problems with respect to a quasi-linear equation of parabolic type. Intense interest to the solution to these problems is arisen by requirements of practice as a result of the necessity of taking into account the effects of non-stationary state and of the nonlinearity of heat-mass exchange processes.

As a result of this, the development of efficient computational algorithms for the solution to specific applied problems is of considerable interest at present.

In the work, we consider an optimal control problem with respect to a quasi-linear heat conduction equation with two control parameters. Analytical formulas for the gradient of the functional are obtained. With the help of these formulas, an algorithm of the numerical solution to the problem using finite difference method on a non-uniform mesh is developed.

Consider a heat conduction process the mathematical model of which has the following form:

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+f(t)=c(u)\frac{\partial u}{\partial t},\quad (x,t)\in G=\{0<x<l,\ 0<t\le T\}, \tag{1}$$

$$u(x,0)=\varphi(x),\quad 0\le x\le l, \tag{2}$$

$$u(0,t)=v(t)\quad 0\le t\le T, \tag{3}$$

$$u(l,t)=g(t)\quad 0\le t\le T, \tag{4}$$

where $k(u)$, $c(u)$, $g(t)$ are given functions; l, T are given values.

In the framework of model (1)-(4), the following problem is set: to find functions $f(t)$, $v(t)$, and $u(x,t)$ satisfying conditions (1)-(4), constraints

$$\left.\begin{array}{l} f_{\min}\le f(t)\le f_{\max}, \\ v_{\min}\le v(t)\le v_{\max}, \end{array}\right\}\quad 0\le t\le T, \tag{5}$$

such that at given function $u^*(x)$ the functional

$$J(f,v)=\int_0^l \left(u(x,T)-u^*(x)\right)^2 dx \tag{6}$$

would take on its minimal possible value, where $f_{\min}$, $f_{\max}$, $v_{\min}$, $v_{\max}$ are known values.

To the numerical solution to problem (1)-(6), we apply conditional gradient method, which requires deriving formulas for the gradient of the functional.

Let $f(t), v(t)$ be arbitrary admissible controls, and $u(x,t)$ is the corresponding solution to boundary problem (1)-(4). Give arbitrary admissible increments $\Delta f(t)$ and $\Delta v(t)$ to the functions $f(t)$ and $v(t)$. Then, $u(x,t)$ obtains the increment $\Delta u(x,t)$ satisfying the following relations:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(k(u+\Delta u)\dfrac{\partial (u+\Delta u)}{\partial x}\right) + f(t) + \Delta f(t) = c(u)\dfrac{\partial (u+\Delta u)}{\partial t}, & 0 < x < l,\ 0 < t \le T, \\ u(x,0) + \Delta u(x,0) = \varphi(x), & 0 < x < l, \\ u(0,t) + \Delta u(0,t) = v(t) + \Delta v(t), & 0 \le t \le T, \\ u(l,t) + \Delta u(l,t) = g(t), & 0 \le t \le T. \end{cases} \tag{7}$$

Applying Taylor formula and taking into account only first order terms, we obtain:

$$\begin{gathered} k(u+\Delta u) = k(u) + \frac{\partial k}{\partial u}\Delta u + o(\Delta u), \\ c(u+\Delta u) = c(u) + \frac{\partial c}{\partial u}\Delta u + o(\Delta u), \end{gathered} \tag{8}$$

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(k(u)\Delta u\right) + f(t) + \Delta f(t) = \frac{\partial u}{\partial t} + \frac{\partial}{\partial t}\left(c(u)\Delta u\right) + o(\Delta u). \tag{9}$$

Taking into account (1)-(4), from (7) and (9), we obtain the following problem fro the increment Δu :

$$\begin{cases} \dfrac{\partial^2}{\partial x^2}\left(k(u)\Delta u\right) + \Delta f = \dfrac{\partial}{\partial t}\left(c(u)\Delta u\right), & 0 < x < l, \quad 0 < t \le T, \\ u(x,0) = 0, & 0 \le x \le l, \\ \left.\begin{aligned} &\Delta u(0,t) = \Delta v(t), \\ &\Delta u(l,t) = 0, \end{aligned}\right\} & 0 \le t \le T. \end{cases} \tag{10}$$

At that functional (6) obtains the following increment

$$\Delta J(f,v) = 2\int_0^l \left(u(x,T) - u^*(x)\right)^2 dx + \int_0^l \left(\Delta u(x,T)\right)^2 dx \ . \tag{11}$$

Multiply the first equation of (10) by $y(x,t)$ and integrate over the domain $\overline{G} = \{0 \le x \le l,\ 0 \le t \le T\}$:

$$\begin{gathered} \int_0^T \frac{\partial}{\partial x}\left(k(u)\Delta u\right) y\Big|_0^l dt - \int_0^T k(u)\Delta u \frac{\partial y}{\partial x}\Big|_0^l dt + \int_0^T k(u)\Delta u \frac{\partial y}{\partial x}\Big|_0^l dt + \int_0^T\int_0^l k(u)\frac{\partial^2 \Psi}{\partial x^2}\Delta u dx dt + \\ + \int_0^T\int_0^l y(x,t)\Delta f(t) dx dt = \int_0^l c(u)\Delta u \cdot y\Big|_0^T dt - \int_0^T\int_0^l c(u)\frac{\partial y}{\partial t}\Delta u dx dt. \end{gathered} \tag{12}$$

Let $y(x,t)$ be the solution to the boundary problem:

$$\begin{cases} k(u)\dfrac{\partial^2 y}{\partial x^2} + c(u)\dfrac{\partial y}{\partial t} = 0, & 0 < x < l, \quad 0 \le t < T, \\ y(x,T) = \dfrac{2(u(x,T) - u^*(x))}{c(u(x,T))}, & 0 \le x \le l, \\ y(0,t) = y(l,t) = 0, & 0 \le t \le T. \end{cases}$$

Taking into account (10) and (13), from (12), we obtain:

$$2\int_0^l \left(u(x,T) - u^*(x)\right)\Delta u(x,T) dx = \int_0^T\int_0^l y(x,t)\Delta f(t) dx dt + \int_0^T k(u(0,t))\frac{\partial y(0,t)}{\partial x}\Delta v(t) dt\ . \tag{13}$$

Then (11) takes on the form

$$\Delta J(f,v)=\int_0^T\int_0^l y(x,t)\Delta f(t)dxdt+\int_0^T k(u(0,t))\frac{\partial y(0,t)}{\partial x}\Delta v dt+\int_0^T(\Delta u(x,T))^2 dx. \quad (14)$$

By definition of the gradient of functional [1]

$$grad\, J(f,v)=(J_f,J_v)=\left(\int_0^l y(x,t)dx,\ k(u(0,t))\frac{\partial y(0,t)}{\partial x}\right). \quad (15)$$

Therefore, the solution to the problem is reduced to finding functions $f(t), v(t), u(x,t), y(x,t)$ from conditions (1)-(5), (13), and (15), for which functional (6) takes on its minimal value.

To find functions $f(t)$ and $v(t)$, we build sequences $f^{(n)}(t)$ and $v^{(n)}(t)$ by formulas

$$\begin{aligned} f^{(n+1)}(t) &= f^{(n)}(t)+\alpha_n\left(\overline{f}^{(n)}(t)-f^{(n)}(t)\right), \\ v^{(n+1)}(t) &= v^{(n)}(t)+\alpha_n\left(\overline{v}^{(n)}(t)-v^{(n)}(t)\right). \end{aligned} \quad (16)$$

Setting admissible initial approximations $f^{(0)}(t)$ and $v^{(0)}(t)$, where

$$f^{(n)}(t)=(f_{\max}+f_{\min})/2+((f_{\max}-f_{\min})/2)signJ_f,$$

$$v^{(n)}(t)=(v_{\max}+v_{\min})/2+((v_{\max}-v_{\min})/2)signJ_v,$$

$$sign\, x=\begin{cases}1, & \text{if } x>0,\\ 0, & \text{if } x=0,\\ -1, & \text{if } x<0,\end{cases}$$

α_n – the step of gradient method, – is selected from the condition of monotone decrease of functional (8) using bisection method [1].

For the numerical implementation of the algorithm, we use difference method on a non-uniform mesh, which is built on the basis of preliminary information on the properties of the solutions.

Boundary problems (1)-(4) and (13) in the presence of fixed $f^{(n)}(t)$ and $v^{(n)}(t)$ are approximated by two-layer implicit difference problems [2, 3], and the schemes obtained are solved using sweep method at each iteration. For approximate computation of the integrals, we use trapezoid quadrature formula. The precision on the functional is checked by the fulfillment of the condition $J^{(n)}-J^{(n+1)}<\varepsilon$, where $\varepsilon>0$ is given small value.

The numerical experiments carried out show the efficiency of the algorithm proposed, and it can be applied to the determination of optimal regimes of heat conduction processes at a long interval of the change of temperature, particularly, of thermo-impact processes in oil reservoirs.

References

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