GREEDY ALGORITHMS ON SPECIAL CONVEX-ORDERED SETS

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In this paper we apply the theory of ordered convexity to convex integer programming. A general methodolologij is developed for worst-case analysis of greedy algorithms.

Let $Z^n = (Z^n, \le)$ $(Z_+^n = (Z_+^n, \le))$ be the set of all (nonnegative) integer $n-$vectors. If $0 = (0,...,0) \in P \subseteq Z_+^n$, P is finite, and the conditions $x \le y$ and $x, y \in P$ imply the inclusion $[x, y] = \{z : x \le z \le y, z \in Z_+^n\} \subseteq P$ then the set P is called a finite ordered-convex set with zero [1]. In what follows, we assume that $P \subseteq Z_+^n$ is a finite ordered-convex set with zero.

A function $f : Z_+^n \to R$ (where R denotes the set of real numbers) is said to be coordinate-convex [1, 2], if

$$\Delta_{ij} f(x) = \Delta_j f(x + e^i) - \Delta_j f(x) \le 0, \forall x \in Z_+^n, i, j \in N = \{1,2,...,n\},$$

where

$$\Delta_j f(x) = f(x + e^j) - f(x), e^j = (e_1^j,...,e_n^j), e_j^j = 1, e_j^k = 0, j \ne k, j,k \in N.$$

A usual, a function $f : Z_+^n \to R$ is no decreasing, if $\Delta_i f(x) \ge 0$ for any $x \in Z_+^n$ and $i \in N$.

Consider the discrete optimization problem (which we refer to as Problem A)

$$\max\{ f(x) : x = (x_1,...,x_n) \in P_\psi \},$$

where $f : Z_+^n \to R$ is a nondecreasing coordinate-convex function, $P_\psi = \{x \in P : \psi(x) = (Ax, x)/2 - b \le 0, P \subseteq Z_+^n$ - ordered- convexity set, $A = (a_{ij})_{n\times n} \in R^{n\times n}, a_{ij} \ge 0, a_{ij} = a_{ji}$ for $(i, j) \in N \times N, x \in Z_+^n, b \in R, b > 0$. By (Ax, x) we denote the inner product of the vectors Ax and x.

Theorem 1. If $\psi(x)$ is a nondecreasing function and $\psi(x) \in Z^1, \forall x \in Z_+^n$,then set P_ψ is order-convex.

Let x^* be an optimal solution Problem A, and let x^g be its gradient solution, i.e., the point obtained by applying the gradient coordinate ascent algorithm (see. e.g. [1-3]). By a guaranteed error estimate for the gradient algorithm in Problem A we mean a number $\varepsilon \ge 0$ for which

$$\frac{f(x^*) - f(x^g)}{f(x^*) - f(0)} \le \varepsilon .$$

Denote by $\lambda = (\lambda_1,...,\lambda_n)$ the characteristic vectors of matrix A, and $\lambda(A) = \max\{\lambda_i : i \in N\}$ - spectral radii of matrix A.

Theorem 2. If $f(x)$ in Problem A is a nondecreasing function on the set $P \subseteq Z_+^n$, $0 < \lambda(A) \le 2/(2h+1)$, the gradient algorithm for solving Problem A has the guaranteed error estimate

$$\varepsilon = 1 - \frac{(2h+1)\lambda(A)}{2 + (2h+1)\lambda(A)},$$

where $h = \max\{x_1 + ... + x_n : x = (x_1,...,x_n) \in P_\psi\}$.

Corollary. Let $\lambda(A) = 2/(2h+1)$. Then under the assumptions of theorem 2, the gradient algorithm for solving Problem A has the guaranteed error estimate

$$\varepsilon = \frac{1}{2}.$$

References

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