OPTIMAL CONTROL PROBLEM FOR THE NONLINEAR PARABOLIC SYSTEM WITH FIXED FINAL STATE

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We consider an open bounded set Ω of Euclid space $\mathbf{R}^n$ with boundary S, $T > 0$, $Q = \Omega \times (0,T)$, $\Sigma = S \times (0,T)$. The system is described by the equation

$$\frac{\partial y}{\partial t} - \Delta y + a(x,t;y) = v,\ (x,t) \in Q\,, \tag{1}$$

with boundary conditions

$$y = 0,\ (x,t) \in \Sigma\,, \tag{2}$$

$$y(x,0) = 0,\ x \in \Omega\,. \tag{3}$$

Let a be a Carathéodory function with inequalities

$$\left|a(\xi;\varphi)\right| \le a_0(\xi) + b\,|\,\varphi\,|^{q-1},$$

$$a(\xi;\varphi)\varphi \ge c\,|\,\varphi\,|^{q},$$

$$[a(\xi;\varphi) - a(\xi;\psi)](\varphi - \psi) \ge 0$$

for all $\varphi,\psi \in \mathbf{R}$ and a.e. on Q, where

$$a_0 \in L_{q'}(Q),\ b > 0,\ c > 0,\ q > 1.$$

We can note as example the function

$$a(s,\varphi) = |\,\varphi\,|^{q-2}\ \varphi,$$

which satisfies to these properties.

We definite the space

$$Y = \left\{ y \mid y \in X,\ y' \in X' \right\},$$

where

$$X = L_2\left(0,T;H_0^1(\Omega)\right) \cap L_q(Q).$$

The space X' is its conjugate. It definite by equality

$$X' = L_2\left(0,T;H^{-1}(\Omega)\right) + L_{q'}(Q),$$

where $1/q + 1/q' = 1$. Then for all control v from the space $V = X'$ the problem (1) – (3) has a unique solution $y = y[v]$ from Y by the known theory of nonlinear parabolic equations [1].

The set U_d of admissible controls is defined by such functions v from convex closed subset U of the space V, which guaranties the equality

$$y(x,T) = w,\ x \in \Omega, \tag{4}$$

where w is the known function from the space $L_2(\Omega)$. We suppose that this set is nonempty. The minimizing functional is determined by equality

$$I(v) = \frac{\alpha}{2}\int_Q v^2 dQ + \int_Q F\left(x,t;y[v],\nabla y[v]\right)dQ,$$

where $\alpha > 0$, F is Carathéodory function on the set $Q \times \mathbf{R}^{n+1}$, besides $F(\xi;\varphi,\cdot)$ is convex and satisfies to the inequality

$$F(\xi;\varphi,\psi) \ge \Phi\left(|\psi|\right)$$

for all $\varphi \in \mathbb{R}$, $\psi \in \mathbb{R}^n$, a.e. on $\xi \in Q$. Here the increasing convex function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies to the condition

$$\Phi(\sigma)/\sigma \to \infty \text{ if } \sigma \to \infty.$$

Problem *P*. *Minimization of functional I for the system* (1) – (3) *on the set* U_d.

Optimal control problems for linear parabolic systems with fixed final state (see [2] – [4]) or for nonlinear parabolic systems with infixed final state (see [5] – [7]) are well known. However we have the nonlinear parabolic systems with fixed final state.

Theorem 1. *The Problem P is solvable*.

Let *u* be a solution of this problem. By penalty method we definite the functional

$$I_k(v) = I(v) + \frac{1}{2\varepsilon_k}\int_\Omega \left| y[v](x,T) - w(x)\right|^2 dx,$$

where $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ if $k \to \infty$. We obtain the regularization problem P_k of minimizing the functional I_k for the system (1) – (3) on the set *U*. This problem has a solution u_k. Let y_k be the corresponding solution of the system (1) – (3).

Theorem 2. *If* $k \to \infty$ *we obtain the convergences* $u_k \to u$ *weakly in V*, $y_k(T) \to w$ *weakly in* $L_2(\Omega)$ *and* $I(u_k) \to \min I(U_d)$.

This result can be use for the finding of the approximate solution of the given problem.

Definition 1. *A control* $v \in U$ *is the* ***approximate solution*** *of the Problem P, if the inclusions* $v \in u + O$, $y(v;T) \in w + O'$
and the inequality

$$I(v) \le \min I(U_d) + \delta$$

are true for small enough neighborhoods in weak topology of zeros O in V and O' *in* $L_2(\Omega)$ *and small enough number* $\delta > 0$.

The control u_k is the approximate solution of the Problem *P* for the large enough number *k* by Theorem 2. Then we have to solve the problem P_k for the large enough number *k*. The substantiation of the corresponding necessary conditions of optimality requires to the differentiability of the solution of our state system with respect to the control in the point *u*. However we have the following results.

Theorem 3. *The mapping* $y[\cdot]: V \to Y$ *for the system* (1) – (3) *is not Gateaux differentiable for larges enough values of parameters q and n.*

This circumstance interrupts to use the known methods of the resolution of optimal control problems for nonlinear parabolic equations (see, for example, [5] – [7]). Therefore we will use the extended differentiability [8]. It is the weaker property than Gateaux differentiability.

Definition 2. *An operator* $A: V \to Y$ *is* $(V_*, Y_*; V_0, Y_0)$***-extended differentiable*** *in the point u, if it exists Banach spaces* V_*, Y_*, V_0, Y_0 *with continuously inclusions* $V_* \subset V_0 \subset V$, $Y \subset Y_0 \subset Y_*$ *and linear continuous operator* $D: V_0 \to Y_0$ *such as*

$$\frac{A(u+\sigma h) - Au}{\sigma} \to Du \quad in\ Y_*$$

for all $h \in V_*$ *if* $\sigma \to 0$.

We suppose that the function $a(\xi;\cdot)$ has the derivative $a_y(\xi;\varphi)$ for all $\varphi \in \mathbb{R}$, which is the Carathéodory function with inequality

$$\left|a_y(x,y)\right| \le a'(x) + b' \,|\, y\,|^{q-2} \;\; \forall x \in \Omega, y \in \mathbb{R},$$

where $a' \in L_{q/(q-2)}(Q),\ b' > 0.$ We determine the spaces

$$Y_* = L_2(0,T;H_0^1(\Omega)),\ V_* = Y_*',\ \ Y_0 = \left\{ p \,\middle|\, p \in Y_*,\ g_0 p \in L_2(Q) \right\},\ V_0 = Y_0',$$

where

$$g_0 = g_0(\xi) = \sqrt{a_y\left(\xi; y[u](\xi)\right)}.$$

Let $p[\mu]$ be the solution of the boundary problem

$$-\frac{\partial p[\mu]}{\partial t} - \Delta p[\mu] + a_y\left(x,t;y[u]\right) p[\mu] = \mu_Q,\ (x,t) \in Q, \tag{5}$$

$$p[\mu] = 0,\ (x,t) \in \Sigma, \tag{6}$$

$$p[\mu](x,T) = \mu_\Omega,\ x \in \Omega, \tag{7}$$

where $\mu = (\mu_Q, \mu_\Omega)$.

Theorem 4. *The mapping* $y[\cdot]: V \to Y$ *for the problem* (1) – (3) *has the* $(V_*, Y_*; V_0, Y_0)$ *- extended derivative* $y'[v]$ *in the arbitrary point* $u \in V$ *. Besides it exists the linear continuous operator* $y_T'[u]: V \to L_2(\Omega),$ *such as*

$$\left\{ y[u+\sigma h](T) - y[u](T) \right\} / \sigma \to y_T'[u]h \ \text{ in } \ L_2(\Omega)$$

if $\sigma \to 0$ *. We have also the equality*

$$\int_Q \mu_Q y'[u]h dQ + \int_\Omega \mu_\Omega y_T'[u]h d\Omega = \int_Q p[\mu]h dQ \ \ \forall h \in V_0,\ \mu_Q \in X_0',\ \mu_\Omega \in L_2(\Omega).$$

We suppose that the derivative $F(\xi;\cdot)$ has the continuous partial derivatives $F_0'(\xi;\zeta)$, $F_1'(\xi;\zeta),\ \ldots,\ F_n'(\xi;\zeta)$ for $\zeta \in \mathbb{R}^{n+1}$, besides the following inequalities

$$\left|F(\xi;\zeta)\right| \le a_F(\xi) + b_F \sum_{i=0}^{n+1} \left|\zeta_i\right|^2,\ \ \left|F_j'(\xi;\zeta)\right| \le a_j(\xi) + b_j \sum_{i=0}^{n+1} \left|\zeta_i\right|,\ j = 0,\ldots,n+1,$$

are true, where

$$a_F \in L_1(Q),\ a_j \in L_2(Q),\ b_F > 0,\ b_j > 0,\ j = 0,\ldots,n+1.$$

We have the possibility to prove the differentiability with respect to the subspace [9] of the regularizing functional now.

Theorem 5. *The functional* I_k *is differentiable with respect to the subspace* V_* *in the arbitrary point* $u_k \in V$, *besides its derivative satisfies to the equality*

$$I_k'(u_k) = p_k + \alpha u_k,$$

where p_k *is the solution of the boundary problem*

$$-\frac{\partial p_k}{\partial t} - \Delta p_k + a_y\left(x,t;y_k\right) p_k = F_0'\left(x,t;y_k,\nabla y_k\right) - \operatorname{div} F_\nabla'\left(x,t;y_k,\nabla y_k\right),\ (x,t) \in Q, \tag{8}$$

$$p_k = 0,\ (x,t) \in \Sigma, \tag{9}$$

$$p_k(x,T) = \left[y_k(x,T) - w(x) \right] / \varepsilon_k,\ x \in \Omega, \tag{10}$$

and $y_k = y[u_k]$.

It is known that the functional, which are differentiable with respect to the subspace V_* of the space *V*, has the minimum on the convex close subset *U* from *V* in the point *u*, if the variational inequality

$$\left\langle J'(u), v-u \right\rangle \geq 0 \ \forall v \in U(u)$$

is true, where $\left\langle \lambda, v \right\rangle$ is the value of a linear continuous functional λ in the point v. The set $U(u)$ consists of all points $v \in U$, satisfying to the inclusion $(v-u) \in V_*$. Then we obtain the necessary condition of optimality for the problem P_k by means of the Theorem 5.

Theorem 6. *The solution of the problem P_k satisfies to the variational inequality*

$$\int\limits_Q \left(\alpha u_k + p_k \right)(v-u_k)dQ \ \forall v \in U(u_k). \tag{11}$$

Thus the solution of the regularization problem can be finding from the system, which includes the boundary problem (1) – (3) for $v = u_k$, the conjugate system (8) – (10) and the variational inequality (11). It becomes the approximate solution of the initial optimal control problem for large enough value k.

We give as addition a result of the controllability of our system.

Definition 3 [10]. *The system* (1) – (3) *is* ***approximate controllable*** *if the set*

$$Y_T = \left\{ y[v](T) \middle| v \in V \right\}$$

is dense in the space $L_2(\Omega)$.

Theorem 7. *We suppose that the following condition*

$$n \leq 2 \ or \ q \leq 2n/(n-2) \ for \ n \geq 3 \ .$$

are true. Then the system (1) – (3) *is approximate controllable.*

This result gives some information about the set U_d.

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