HOMOGENEOUS GROUPINGS IN PORTFOLIO MANAGEMENT

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Abstract: Often, in situations of uncertainty in portfolio management, it is difficult to apply the numerical methods based on the linearity principle. When this happens it is possible to use nonnumeric techniques to assess the situations with a non linear attitude. One of the concepts that can be used in these situations is the concept of grouping. In the last thirty years, several studies have tried to give good solutions to the problems of homogeneous groupings. For example, we could mention the Pichat algorithm, the affinities algorithms and several studies developed by the authors of this work. In this paper, we use some topological axioms in order to develop an algorithm that is able to reduce the number of elements of the power sets of the related sets by connecting them to the sets that form the topologies. We will apply this algorithm in the grouping of titles listed in the Stock Exchange or in its dual perspective.

1. The two perspectives for topological fuzzification in economy

It is well known that a topology E in uncertainty can be defined by the subset T(E) of the opened that accomplishes the following axioms (Chang, 1968). Note that for further reading on fuzzy topology and pretopology, we recommend, for example (Badard, 1981; Bayoumi, 2005; Du *et. al*., 2005; Fang and Chen, 2007; Fang and Yue, 2004; Gil-Aluja, 2003; Gil-Aluja and Gil-Lafuente, 2007; Saadati and Park, 2006; Yan and Wu, 2007; Yue, 2007):

1. $\phi \in T(E)$
2. $E \in T(E)$
3. $(\underset{\sim}{A}_j \in T(E), \underset{\sim}{A}_k \in T(E)) \rightarrow (\underset{\sim}{A}_j \cap \underset{\sim}{A}_k \in T(E))$
4. $(\underset{\sim}{A}_j \in T(E), \underset{\sim}{A}_k \in T(E)) \rightarrow (\underset{\sim}{A}_j \cup \underset{\sim}{A}_k \in T(E))$
5. $(\underset{\sim}{A}_j \in T(E)) \rightarrow (\overline{\underset{\sim}{A}}_j \in T(E))$

where $\underset{\sim}{A}_j, \underset{\sim}{A}_k$ may have a different meaning depending on the criteria used for the fuzzification.

In the first case, they are fuzzy subsets of the referential set E that accomplishes the previous axioms, and in the other case, they are elements of the power set established from a referential set E of fuzzy subsets. In the first case, the referential set E is formed initially by the elements of the referential set of the fuzzy subsets. In the second case, their elements are the fuzzy subsets theirselves. As it has been pointed out in other works (Gil-Aluja, 2003; Gil-Aluja and Gil-Lafuente, 2007) the selection of one of these perspectives depends mainly on the objectives of the analysis. In an economical and financial context, we consider that it is relevant to think about the meaning of the components of both cases. For doing this, we will use the representability of the notion of fuzzy subset. The reason is because for an economist, a fuzzy subset is a descriptor of a physical or mental object; and this description is developed by putting different levels to the elements of the referential set formed by the attributes of the objects that we want to describe. Then, in the economic environment it is possible to accept that in the first case, the referential set E is formed by the set of attributes that describe each object while in the second case, the referential set E is formed by the fuzzy subsets, where each of them describe an object. If we consider financial products such as titles listed in the Stock Exchange, the description of each of them will take place by a certain number of attributes such as the expected rentability, the liquidity capacity without loses, etc., all of them classified at certain level. In this assumption, the referential set E will be formed in the first case by the expected rentability, the liquidity capacity, etc., and in the second case, by the different titles listed in the Stock Exchange. With this approach, the $\underset{\sim}{A}_j, \underset{\sim}{A}_k \in T(E)$, the elements of the open set T(E), are in the first case, fuzzy subsets with the referential of their attributes and in the second case, fuzzy subsets or groupings of fuzzy subsets with the same referential. It is obvious that the concept of economic representability is different in each case. Then, the axioms 1 and 2 acquire the following meaning: In the first case, axiom 1 shows that the fuzzy subset (title listed in the Stock Exchange) with a null level in all its attributes is an open set and so is (axiom 2) the fuzzy subset with level one (maximum) in all its attributes. In the second case, axiom 1 shows that in a situation without fuzzy subsets we have an open set. In this case, the set of all the fuzzy sets (all the titles listed in the Stock Exchange) is also an open set (axiom 2). In axiom 3, we also find different meanings depending on the case analyzed: In the first one, axiom 3 requires that if a fuzzy subset with certain levels for each attribute is an open set and so is another fuzzy subset with its own levels, then, there exists a third one that it is also an open set with a membership level for each attribute that is equal to the lowest of the other two. In the second one, we can see that if a group of fuzzy subsets is an open set and so is another group of fuzzy subsets, then, the group of fuzzy subsets that is contained in both groups, is also an open set. Finally, axiom 4 expresses the following for each case: In the first one, if we have a fuzzy subset with certain levels for each attribute and another one with its own levels, and both are open sets, then, there exists another fuzzy subset that it is also an open set. The membership level of the attributes of this fuzzy subset is given by the maximum between the other two fuzzy subsets. In the second one, if we have two groups of fuzzy subsets that are open sets, then, there exists a third one that is also an open set and it comprises the fuzzy subsets of the first and/or the second group. Sometimes, it can be useful to use as open sets, the complementary of any open set. This implies the necessity of considering another axiom as follows:

In this axiom, the representativity also acquires a different meaning depending on the case used. Then: In the first case, it is necessary that if a fuzzy set is an open set, then, the fuzzy subset which has a complimentary level to the first one in all the attributes has also to be an open set. Then, if for a certain attribute an open set has a level α the complimentary fuzzy subset will have 1 - α, where $\alpha \in [0, 1]$. In the second case, when a grouping of fuzzy subsets is an open set, then, the group formed by the rest of fuzzy subsets is also an open set. Focusing in this important context, we believe that it is interesting to note that it is not necessary to establish the existence of the five axioms presented above for arriving to the same result. This happens because if three of the axioms are accomplished, then, the other two will be accomplished automatically. These three axioms are:

1) $E \in T(E)$

2) $(\underset{\sim}{A}_j \in T(E)) \rightarrow (\overline{\underset{\sim}{A}_j} \in T(E))$

3) $(\underset{\sim}{A}_j \in T(E), \underset{\sim}{A}_k \in T(E)) \rightarrow (\underset{\sim}{A}_j \cup \underset{\sim}{A}_k \in T(E))$

As we can see, with the first and the second axiom, it is satisfied: $\phi \in T(E)$

And, due to: $\underset{\sim}{A}_j \cup \underset{\sim}{A}_k \in T(E)$ Is also: $\overline{\underset{\sim}{A}_j \cup \underset{\sim}{A}_k} \in T(E)$

By using De Morgan theorem: $\overline{\underset{\sim}{A}_j \cup \underset{\sim}{A}_k} = \overline{\underset{\sim}{A}}_j \cap \overline{\underset{\sim}{A}}_k$ Then: $\overline{\underset{\sim}{A}}_j \cap \overline{\underset{\sim}{A}}_k \in T(E)$

Now, it is interesting to establish these two general cases in the financial environment and it's representatively in a real problem of the financial operations. Note that in this paper we will only focus on the first case.

2. The hypothesis of a referential set of referentials

We will assume a set of attributes of titles listed in the Stock Exchange that are significant for the potential investors. Assume a referential set E of attributes of titles listed in the Stock Exchange, as follows: $E = \{x_j \,/\, j = 1,2,..., m\}$ where the x_j represent, for example, the expected rentability, the liquidity capacity, etc. Now, we describe each title by using a fuzzy subset of the referential of its attributes that we designate as $A_k, k = 1,2,..., n$, where n indicates the total number of titles considered. Then, each of these titles will be described as follows:

$(E, \mu \underset{\sim}{A}_k(x_j)), \mu \underset{\sim}{A}_k \in [0,1]$

Then, we establish a relation between title and attribute, such that if $x_j \in E$ possess a value of the membership function for $\underset{\sim}{A}_k$ with a level μ, we write it as: $(\mu \underset{\sim}{A}_k(x_j) = \mu, \mu \in [0,1]$

The description of these titles by its attributes permit us to know the expected level of each attribute $x_j / j = 1,2,..., m$, that has been assigned for each title. However, the investor of the titles often establishes, for each attribute, a minimum level or threshold, where he assumes that a level below the threshold can be considered as zero. Then, it is necessary to establish a fuzzy subset of thresholds that we will designate as U where: $\mu \cup (x_j) = \lambda_j \in [0,1]$

This means that we establish α_j - cuts such that: $< \alpha_j \rightarrow 0, \geq \alpha_j \rightarrow 1$

Now, we are able to form the family F that comprises the group of attributes that are possessed, at certain level, by the titles. Note that a key aspect in this process is the assignment of valuations to the subset of thresholds, because depending on these valuations, the family F will be different. Next, we analyze the attributes in order to know if one or more of them are possessed by all the titles or not. This analysis can be done by using all the available intersections between F_i and $\overline{F}_i$. Then, we get the no null intersections. By using the largest number of titles for each attribute, we get an optimization. Then, if we unify all these attributes in all the different ways and we add the empty set, we get a topology. As it is well known, a topology can be represented by using a Boolean lattice. Due to all the axioms commented above are accomplished.

3. The dual approach

Following with the same problem, it is interesting to consider the topology if instead of describing each of the tittles by using fuzzy subsets of the referential of the attributes, we describe each attribute by using the levels they get in the titles. Following the same way than before, we find the family of titles, formed by the subset of the ones that possess the attributes x_j. We get the desired topology considering these elements and all their possible unions, and adding the empty set. We can represent with a circle this topology inside a Boolean lattice of the power set.

4. Relation between the two topologies

Now, we go back to the beginning to present the information related to the descriptors of the titles, once developed the adjustment with thresholds. Note that now we will develop the analysis using the matrix form. Next, we relate each of the elements of the topology $T(E_2)$ with those elements of the power set of E_1, the titles that all of them possess and the attributes that each element of the topology $T(E_2)$ establishes. All the groups of titles found are part of the topology $T(E_1)$ and it is possible to present the correspondences represented in the two lattices formed by the topologies $T(E_1)$ and $T(E_2)$. In each vertex of the lattice corresponding to the topology $T(E_1)$ it arrives two lines that come from two vertexes of the lattice of the topology $T(E_2)$. In the other hand, when it has been established the correspondence between the elements of the topology $T(E_2)$ it has been selected the group of elements of the power set of E_2, the group with a larger number of attributes, excluding those groups formed by a smaller quantity. What it has been previously developed visually, now it can be found automatically by choosing in each vertex of the lattice of the topology $T(E_1)$, the line that conducts to the vertex of the lattice of the topology $T(E_2)$ with a larger number of elements, that is, attributes. From another perspective, if we consider all the elements of the topology $T(E_1)$, that is, titles and groups of titles, and the subsets of attributes possessed by all of them are found visually in the matrix. If a topology $T(E_1)$ has a lower number of elements than other topology $T(E_2)$, the correspondence of each element of $T(E_1)$ goes to only one element of $T(E_2)$. This is interesting in order to develop an algorithm for groupings. At least, there exists a correspondence between both topologies. In the business and in the economic environment, it becomes interesting to analyze this type of correspondence, especially when we want to establish homogeneous groupings or segmentation processes. The same result is found by using one of the algorithms used for obtaining affinities (Kaufmann and Gil-Aluja, 1991). But then, it is necessary to consider, in the relation between the set of titles and the set of attributes, all the elements of the power set of both sets. Then, starting from the same matrix and using the algorithm of the maximum inverse correspondence (Gil-Aluja, 1999) we find the right connection. By choosing for all the elements of the power set E_2 the element of the power set of E_1 with the largest number of components. As we can see, the result is the same as the one found with the new algorithm. Sometimes, the topologies $T(E_1)$ and $T(E_2)$ are the same than the power set of both sets E_1 and E_2, as it happens in the following problem (Gil-Aluja and Gil-Lafuente, 2007). Note that this result is the same than the result that appears in the work (Gil-Aluja and Gil-Lafuente, 2007) that uses an assignment algorithm.

5. Algorithm for the maximum relations between the titles and the attributes

Once it has been shown the relation between the two topologies $T(E_1)$ and $T(E_2)$, and the maximizing character of their construction, we will now develop an algorithm that permits to solve the problems of relation between the titles and the attributes. Then, this algorithm will be also able to solve the problem of homogeneous groupings between them with an optimizing character. For doing so, we suggest the following steps:

A- <u>Obtention of the topology $T(E_2)$</u>

1. We describe the titles with fuzzy subsets of the referential of its attributes.
2. We establish a threshold subset of the same referential set of attributes, from where we can obtain the description of the titles by using fuzzy subsets.
3. We form the family F of sets of attributes, where each one of these elements comprises the attributes possessed by each title. This family will comprise as much elements as titles it has.
4. For each element of the family F we get its complementary. We will also find as much complimentary as titles, which will form the family $\overline{F}$.
5. We develop all the possible intersections between the elements of the family F and the elements of the family $\overline{F}$. We select the non empty intersections.
6. These non empty intersections, (each one of them contains one or more attributes) are connected in all the possible ways and we add the empty set. Then, we get a topology $T(E_2)$.

B- <u>Obtention of the topology $T(E_1)$</u>

7. The attributes are described by using fuzzy subsets of the referential set of the titles.
8. Using the same threshold subset, we get the description of each attribute by using Boolean subsets.
9. We form the family of subsets of tittles, the family of complimentaries and we develop all the possible intersections between them in a similar way as described in steps 3, 4 and 5.
10. We select the non empty intersections (each of them contain at least one title), we develop all the possible unions and we add the empty set, obtaining the topology $T(E_1)$.

<u>C- Establishment of relations between two topologies</u>

11. We go back to the beginning to establish the Boolean relations between titles and attributes, by using a graph in order to obtain a better visualization.
12. We select between the two topologies the one with a lowest number of elements in the topology $T(E_1)$. Each element of this topology will be related with the elements of the power set of the other set, in our case E_2 (the attributes), that are possessed by all the titles. We can prove that each of the groups of attributes found are part of the topology $T(E_2)$.
13. In the case that one element of the topology $T(E_1)$ is related with more than one element of the topology $T(E_2)$, then, we select the element with a largest number of attributes.

14. The relation obtained constitutes an optimal that, moreover, permits the grouping of titles with the highest number of shared attributes. It also permits the grouping of attributes that are possessed together by the highest number of titles.

6. Conclusions

The operators that work in the financial markets usually try to form groups of titles that permit to obtain some qualities in order to be prepared against an unexpected volatility. It is obvious that there exist a wide range of methods for selecting the titles of a portfolio. In this paper, we have considered a scenario where we form in a first stage, subsets of titles, where each one of them is constituted by elements that possess certain homogeneity according to some attributes previously established. Note that the objective is to extract one or several titles of each subset in order to get the desired properties. The theoretical solution of an approach as the one described here, has generalized other previous algorithms that were not so complete as this one such as the Pichat algorithm or the Kaufmann and Gil-Aluja algorithm. We believe that the presented algorithm solves the problems of the Pichat and the Kaufmann and Gil-Aluja algorithms. Note that in some exceptional situations the algorithm presented here can become the Kaufmann and Gil-Aluja algorithm. This happens when the two topologies used in the relation are equal to the power sets of the sets of titles and its attributes. In the beginning of this work we have shown two methods for fuzzifying the topologies. The approach developed in this paper is based on one of these methods. Note that in future research, we will consider the possibility of using the other method in the problem. Finally, we want to mention the usefulness in the operations of our algorithm that has been tested with satisfactory results twice with the examples shown in the paper.

References

1. Badard, R., (1981) Fuzzy pre-topological spaces and their representation. *Journal of Mathematical Analysis and Applications*, *81*, 211-220.
2. Bayoumi, F., (2005) On initial and final L-topological groups. *Fuzzy Sets and Systems*, *156*, 43-54.
3. Chang, C. L., (1968). Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, *24*, 182-190.
4. Du, S. H., Qin, Q. M., Wang, Q., Li, B., (2005). Fuzzy description of topological relations I: A unified fuzzy 9-intersection model. In: *Advances in Natural Computation. Pt 3, Proceedings*. Berlin: Springer-Verlag, Berlin, pp. 1261-1273.
5. Fang, J. M., Chen, P. W., (2007). One-to-one correspondence between fuzzifying topologies and fuzzy preorders. *Fuzzy Sets and Systems*, *158*, 1814-1822.
6. Fang, J. M., Yue, Y. L., (2004). K. Fan's theorem in fuzzifying topology. *Information Sciences*, *162*, 139-146.
7. Gil-Aluja, J., (1999). *Elements for a theory of decision under uncertainty*, Kluwer Academic Publishers, Dordrecht.
8. Gil-Aluja, J. (2003). Clans, affinities and Moore's fuzzy pretopology. *Fuzzy Economic Review*, *8*, 3-24.
9. Gil-Aluja, J., Gil-Lafuente, A. M., (2007). *Algoritmos para el tratamiento de fenómenos económicos complejos* (In Spanish). Ed. CEURA, Madrid.
10. Kaufmann, A., Gil-Aluja, J., (1991). Selection of affinities by means of fuzzy relation and Galois lattices. *Proceedings of the XI EURO Congress*, Aachen, Germany.
11. Saadati, R., Park, J. H., (2006). On the intuitionistic fuzzy topological spaces. *Chaos, Solitons & Fractals*, *27*, 331-344.
12. Yan, C. H., Wu, C. X., (2007). Fuzzifying vector topological spaces, *International Journal of General Systems*, *36*, 513-535.
13. Yue, Y. L., (2007). Lattice-valued induced fuzzy topological spaces. *Fuzzy Sets and Systems*, *158*, 1461-1471.