LIMIT SUBCRITICAL BRANCHING PROCESS WITH IMMIGRATION

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Let $\{\xi_{k,j}, k, j \in N\}$ and $\{\varepsilon_k, k \in N\}$ are two independent sets of independent, non-negative integer-valued and identically distributed random variables. Branching process with immigration $\{X_k, k \geq 0\}$ we will define following recurrently as

$$X_0 = 0, \quad X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, k \in N . \tag{1}$$

Let's assume, that, $E\xi_{1,1}^2 < \infty$, $E\varepsilon_1^2 < \infty$ and we will put

$$m = E\xi_{1,1},\ \sigma^2 = \mathrm{var}\,\xi_{1,1},\ \lambda = E\varepsilon_1,\ b^2 = \mathrm{var}\,\varepsilon_1 .$$

Branching process (1) is called subcritical, critical and supercritical, if $m < 1$, $m = 1$ and $m > 1$ respectively. Conditions of weak convergence finite-dimensional distributions of branching processes with immigration have been investigated in [1] and [2]. In [3] it is proved, that step function of a critical branching process with immigration converges in Skorokhod topology to a nonnegative diffusion process.

Let now the immigration stream is nonhomogeneous , that is ε_k, $k \geq 1$ have various distributions. We will assume, that $\lambda_k = E\varepsilon_k$, $b_k^2 = \mathrm{var}\,\varepsilon_k$ are finite for any $k \in N$ and

$$\lambda_n = n^\alpha L_\alpha(n) \text{ as } n \to \infty, \tag{2}$$

$$b_n^2 = n^\beta L_\beta(n) \text{ as } n \to \infty, \tag{3}$$

where $\alpha, \beta \geq 0$ and $L_\alpha(n)$, $L_\beta(n)$ are slowly varying functions on infinity.

In the paper [4] functional limiting theorems for fluctuation critical branching process with immigration in a case when conditions (2) and (3) are satisfied are proved.

In this paper it is investigated asymptotical behavior of X_n in the subcritical a case when immigration satisfies the conditions (2) and (3). Comparison of the our result with corresponding result of [4] shows, that asymptotical behavior critical and subcritical branching processes with immigration essentially differ from each other even in a case when the stream of immigrations with time growth infinitely increases.

Theorem. Let $m < 1$, conditions (2) and (3) are satisfied, and $\lambda_n \to \infty$, $b_n^2 = o(\lambda_n)$ as $n \to \infty$. Then

$$Ee^{it\frac{X_n - EX_n}{\sqrt{\lambda_n}}} \to e^{-\frac{\sigma^2 t^2}{(1-m)(1-m^2)}} \quad \text{as } n \to \infty$$

Proof. From (1) we have

$$X_k = mX_{k-1} + \lambda_k + S_k + \varepsilon_k - \lambda_k , \tag{4}$$

where $S_k = \sum_{j=1}^{X_{k-1}} \left(\xi_{k,j} - m\right)$. Averaging (4) we have

$$EX_k = mEX_{k-1} + \lambda_k . \tag{5}$$

Solving this equation, we receive

$$EX_k = \sum_{j=1}^{n} m^{k-j}\lambda_j . \qquad (6)$$

From (2) and (3) we can assume that λ_n and b_n^2 is monotonously increasing. We will put $r_n = \left[\sqrt{n}\right]$, where $[a]$ denotes the integer part of number a . We have

$$EX_n = \sum_{k=1}^{n-r_n} m^{n-k}\lambda_k + \sum_{k=n-r_n+1}^{n} m^{n-k}\lambda_k . \qquad (7)$$

Clearly, that

$$\sum_{k=1}^{n-r_n} m^{n-k}\lambda_k \le \lambda_{n-r_n} m^{r_n}\frac{1-m^{n-r_n}}{1-m} . \qquad (8)$$

Further we have

$$\lambda_{n-r_n+1}\frac{1-m^{r_n}}{1-m} \le \sum_{k=n-r_n+1}^{n} m^{n-k}\lambda_k \le \lambda_n\frac{1-m^{r_n}}{1-m} . \qquad (9)$$

Now from (6) - (9), considering properties of slowly varying functions, we receive

$$\lambda_n^{-1}EX_n \to \frac{1}{1-m} \text{ as } n\to\infty . \qquad (10)$$

Similar reasoning's, as well as above, lead to a relation

$$\operatorname{var}\frac{X_n}{\lambda_n} \le \frac{\sigma^2}{\lambda_n(1-m)(1-m^2)} + \frac{b_n^2}{\lambda_n^2(1-m^2)} \to 0 .$$

From here and from (10), using Chebyshev inequality we obtain

$$\frac{X_n}{\lambda_n}\xrightarrow{P}\frac{1}{1-m} \text{ as } n\to\infty . \qquad (11)$$

Further, from (4) and (5) we have

$$X_k - EX_k = m\left(X_{k-1} - EX_{k-1}\right) + S_k + \varepsilon_k - \lambda_k .$$

Hence,

$$X_n - EX_n = \sum_{k=1}^{n} m^{n-k}M_k = \sum_{k=1}^{n} m^{n-k}S_k + \sum_{k=1}^{n} m^{n-k}\left(\varepsilon_k - \lambda_k\right) . \qquad (12)$$

Since the random variables ε_k, $k\ge 1$ are independents and $b_n^2 = o\left(\lambda_n\right)$, we have

$$\operatorname{var}\frac{1}{\sqrt{\lambda_n}}\sum_{k=1}^{n} m^{n-k}\left(\varepsilon_k - \lambda_k\right) = \frac{1}{\lambda_n}\sum_{k=1}^{n} m^{2(n-k)}b_k^2 \le \frac{b_n^2}{\lambda_n}\cdot\frac{1-m^{2n}}{1-m^2} \to 0 \qquad (13)$$

as $n\to\infty$. We consider

$$\sum_{k=1}^{n} m^{n-k}S_k = \sum_{k=1}^{n-r_n} m^{n-k}S_k + \sum_{k=n-r_n+1}^{n} m^{n-k}S_k . \qquad (14)$$

Applying a known inequality $\left(\sum_{k=1}^{n} a_k\right)^P \le n^{P-1}\sum_{k=1}^{n} a_k^P\,(a_i\ge 0, i=\overline{1,n})$ we receive

$$\operatorname{var}\frac{1}{\sqrt{\lambda_n}}\sum_{k=1}^{n-r_n} m^{n-k}S_k \le \frac{n}{\lambda_n}\sum_{k=1}^{n-r_n} m^{2(n-k)}\operatorname{var}S_k = \frac{n}{\lambda_n}\sum_{k=1}^{n-r_n} m^{2(n-k)}\sigma^2 EX_{k-1} \le \qquad (15)$$

$$\le \frac{n}{\lambda_n}\lambda_{n-r_n}\cdot\frac{\sigma^2}{1-m}\cdot m^{2r_n}\frac{1}{1-m^2} = \frac{\sigma^2}{(1-m)\left(1-m^2\right)}\frac{\lambda_{n-r_n}}{\lambda_n}\cdot nm^{2r_n} \to 0 \text{ as } n\to\infty ,$$

where it is considered, that $nm^{2r_n} \to 0$ as $n\to\infty$.

Let f is characteristic function of $\xi_{1,1}-m$, F_k denote the σ–algebra generated by $\{X_0, X_1, ..., X_k\}$. Using independence of $\xi_{k,j}$, $k, j \in N$ we find that

$$Ee^{it\frac{1}{\sqrt{\lambda_n}}\sum_{k=n-r_n+1}^{n} m^{n-k}S_n} = E\prod_{k=n-r_n+1}^{n} f^{X_{k-1}}\left(\frac{m^{n-k}t}{\sqrt{\lambda_n}}\right) \tag{16}$$

According to Taylor's formula for enough big n we have

$$f\left(\frac{m^{n-k}t}{\sqrt{\lambda_n}}\right) = 1 - \frac{\sigma^2 m^{2(n-k)}}{\lambda_n}t^2 + o\left(\frac{m^{2(n-k)}}{\lambda_n}t^2\right). \tag{17}$$

Now, consistently applying known inequalities

$$\left|e^x - e^y\right| \le \left|x-y\right|, \quad \operatorname{Re} x \le 0, \operatorname{Re} y \le 0 \tag{18}$$

and $\left|\ln(1+x)-x\right| \le x^2$ considering (17), (10), we receive

$$\left|E\prod_{k=n-r_n+1}^{n} f^{X_{k-1}}\left(\frac{m^{n-k}t}{\sqrt{\lambda_n}}\right) - Ee^{-\sum_{k=n-r_n}^{n} X_{k-1}\frac{\sigma^2 m^{2(n-k)}}{\lambda_n}t^2}\right| =$$

$$\le E\sum_{k=n-r_n+1}^{n} X_{k-1}\left|\ln f\left(\frac{m^{n-k}t}{\sqrt{\lambda_n}}\right) - \frac{\sigma^2 m^{2(n-k)}}{\lambda_n}t^2\right| \le \tag{19}$$

$$\le \sum_{k=n-r_n+1}^{n} EX_{k-1}\cdot\frac{\sigma^4 m^{4(n-k)}}{\lambda_n^2}t^4 \le \frac{\sigma^4 t^4}{\lambda_n^2}\sum_{k=n-r_n+1}^{n} m^{n-k}EX_{k-1} \le \frac{\sigma^4 t^4}{\lambda_n}\cdot\frac{1}{(1-m)^2} \to 0 \text{ as } n\to\infty.$$

Clearly, that for any $t \in R$

$$\left|\sum_{k=n-r_n+1}^{n} X_{k-1}\frac{\sigma^2 m^{2(n-k)}}{\lambda_n}t^2 - \frac{\sigma^2}{(1-m)(1-m^2)}t^2\right| \le$$

$$\le\left[\sup_{n-r_n\le k\le n}\left|\frac{X_{k-1}}{\lambda_n} - \frac{1}{1-m}\right|\sigma^2\sum_{k=n-r_n+1}^{n} m^{2(n-k)} + \sigma^2\frac{m^{r_n}}{(1-m)(1-m^2)}\right]t^2. \tag{20}$$

From (10) we obtain that

$$\sup_{n-r_n\le k\le n}\left|\frac{X_{k-1}}{\lambda_n} - \frac{1}{1-m}\right| \xrightarrow{P} 0 \text{ as } n\to\infty. \tag{21}$$

Then from (20), (21) and theorems Lebeguar about convergence we have

$$Ee^{-\sum X_{k-1}\frac{\sigma^2 m^{2(n-k)}}{\lambda_n}t^2} \to e^{-\frac{\sigma^2 t^2}{(1-m)(1-m^2)}} \text{ as } n\to\infty.$$

Now this and from (19), (13) - (16), (12) completes the proof.

References

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