ON AN INVERSE PROBLEM OF THE DETERMINATION OF SWITCHING CONDITIONS IN DISCONTINUOUS SYSTEMS

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We consider a class of inverse problems for dynamical processes described by discontinuous systems of ordinary differential equations. The form of these equations changes depending on whether the state of the process belongs to one or another sub-domain of the state space. Problems of this kind arise while investigating, modeling, identifying, and controlling many technical objects and technological processes taking into account the impact of the mutable environment on the process. These problems have been considered by many authors [4-6]. In contrast to the works implemented before, the present work is mainly dedicated to the identification of switching or discontinuity surfaces.

Formulas providing the components of the gradient of the chosen performance criterion with respect to the identifiable parameters are established in the work. These formulas make it possible to implement first order optimization methods and to obtain numerical solutions to the considered problems.

We assume that the dynamics of the investigated object is described by a discontinuous system of non-linear differential equations with variable structure of the form:

$$\dot{x}(t) = f^l\left(x(t), p^l(t)\right) \text{ with } x(t) \in X^l(t), \quad t \in (0,T], \quad l = 1,2,...,L. \tag{1}$$

Here $x(t) \in R^n$ is the vector designating the state of the process; $p^l(t) \in R^{r_l}$ are the values of the parameters when the state of the process $x(t)$ belongs to $\mathrm{X}^l(t)$; $\mathrm{X}^l(t)$ are corresponding sub-domains (zones) of the space of all possible states of the process X, i.e. $\mathrm{X}^l(t) \subset \mathrm{X} \subseteq R^n$, $l = 1,2,...,L$. Vector-functions $f^l(.,.)$, $l = 1,2,...,L$ are given to within the functional parameters $p^l = p^l(t)$ and are continuously differentiable on all their arguments. The zones of the phase space

$$\begin{aligned} &\mathrm{X}^l(t) = \left\{x \in R^n : g^{l-1}(x,t) > 0, \quad g^l(x,t) \le 0\right\}, \quad l = 2,3,...,L-1, \\ &\mathrm{X}^1(t) = \left\{x \in R^n : g^1(x,t) \le 0\right\}, \quad \mathrm{X}^L(t) = \left\{x \in R^n : g^{L-1}(x,t) > 0\right\}, \end{aligned} \tag{2}$$

are simply connected and defined by their boundaries by means of identifiable and two times continuously-differentiable functions $g(x,t) = \left(g^1(x,t), g^2(x,t),..., g^{L-1}(x,t)\right)$. We assume that the known vector-functions $f^l(.,.)$, $l = 1,2,...,L$ and the unknown vector-function $g(x,t)$ satisfy the conditions:

$$\left\| f^k\left(x, p^k\right)\right\| < m_1, \quad \left\|\nabla f^k\left(x, p^k\right)\right\| < m_2, \quad \left|g^l(x,t)\right| < m_3, \quad \left\|\nabla g^l(x,t)\right\| < m_4 \tag{3}$$

at $t \in [0,T]$ and $x \in X$, $l = 1,2,...,L-1$, $k = 1,2,...,L$, where $m_i, i = 1,2,3,4$ are given positive values,

$$\operatorname{int}\mathrm{X}^{l_1}(t) \cap \operatorname{int}\mathrm{X}^{l_2}(t) = 0, \quad l_1 \ne l_2, \quad l_1, l_2 = 1,2,...,L, \quad \bigcup_{l=1}^{L}\mathrm{X}^l(t) = X. \tag{4}$$

The vector-function $x(t)$, i.e. the solution to (1), is everywhere continuously-differentiable except for the points of time $\bar{t}_l$ when the trajectory hits the discontinuity surface $g^l\left(x(\bar{t}_l), \bar{t}_l\right) = 0$, $l = 1,2,...,L-1$.

Introduce the notations

$$p(t)=\left(p^1(t),...,p^L(t)\right)=\left(p_1^1(t),...,p_{r_1}^1(t),...,p_{r_L}^L(t)\right)\in R^r,\ r=\sum_{l=1}^{L} r_l\ .$$

In real problems, parameters $p^l(t)$ must satisfy some constraints resulting from technical and technological considerations.

In the aim of identifying the unknown parameters, N independent observations have been carried out over the dynamics of the process at different initial conditions:

$$x^i(0)=x_0^i,\quad i=1,2,...,N\ . \tag{5}$$

At that the current state of the process $x(t)$ will depend on its initial condition x_0, on functions $g^l(x,t)$, $l=1,2,...,L-1$, and on the corresponding values of the parameter $p(t)$, i.e. $x(t)=x(t;x_0,p,g)$. It is evident that the initial states of the object x_0^i, $i=1,2,...,N$, the values of the parameter $p(t)$, and the switching surfaces $g^l(x,t)=0$, $l=1,2,...,L-1$ are independent of each other.

There may be observations over some components or over the whole vector of the state of the object at different points of time:

$$x^i\left(t_{ij};x_0^i,p,g\right)=x^{ij},\quad t_{ij}\in(0,T],\quad j=1,2,...,N_i,\quad i=1,2,...,N\ ,$$

particularly at final point of time T

$$x^i\left(T;x_0^i,p,g\right)=x_T^i,\quad i=1,2,...,N\ . \tag{6}$$

Here N_i is the number of points of time at which observations have been carried out over the state of the object with initial condition x_0^i at i th experiment. There may also be observations over the state of the object at different initial conditions at some time intervals:

$$x^i\left(t;x_0^i,p,g\right)=y^{ij}(t),\ t\in\left[\tau_{ij-1},\tau_{ij}\right]\subset[0,T],\quad \tau_{ij-1}<\tau_{ij},\quad j=1,2,...,N_i,\quad i=1,2,...,N\ .$$

Here N_i is the number of time intervals at initial condition x_0^i, at which observations have been carried out over the object. Observations may also be of mixed type, i.e. both pointwise and interval.

In this work, we consider the most frequently occurring case in practice when the identifiable parameters are piecewise constant functions:

$$p^l(t)=p^l=const,\ p^l\in R^{r_l}\ \text{with}\ x(t)\in X^l,\ l=1,2,...,L,\ t\in[0,T]. \tag{7}$$

The case when $p(t)$ is a function of time has been considered by many authors [6]. That is why in this work, we pay comparatively little attention to the identification of the values of the parameter $p(t)$.

The investigated problem then consists in determining $(L-1)$-dimensional vector-function $g(x,t)$ and finite-dimensional vector $p\in R^r$. In case of observations (6) one can use a mean-square performance criterion

$$J(p,g)=\frac{1}{N}\sum_{i=1}^{N} I_i\left(x^i\left(T;x_0^i,p,g\right),p,g\right),\ \ I_i\left(x^i\left(T;x_0^i,p,g\right),p,g\right)=\left\|x^i\left(T;x_0^i,p,g\right)-x_T^i\right\|_{R^n}^2. \tag{8}$$

Then the investigated identification problem is reduced to a problem of parametrical optimal control, consisting in the minimization of functional (8) with respect to (1)–(7).

In the aim of determining the functions $g^l(x,t)$, $l=1,2,...,L-1$, we propose to parameterize them with the help of some known finite system of linearly independent continuously-differentiable functions $\left\{\varphi^i(x,t)\right\}$, $i=1,2,...,\bar{\nu}$ using the representations of the functions $g^l(x,t)$, $l=1,2,...,L-1$ in the form

$$g^l(x,t) = g^l(x,t;\alpha^l) = \sum_{i=1}^{\nu^l} \alpha_i^l \varphi^i(x,t), \quad l = 1,2,...,L-1, \ \alpha^l \in R^{\nu^l}, \ \nu = \sum_{l=1}^{L-1} \nu^l, \ \bar{\nu} = \max_{1 \le l \le L-1} \nu^l,$$

$$\alpha = \left(\alpha_1^1,...,\alpha_{\nu^1}^1,...,\alpha_{\nu^{L-1}}^{L-1}\right) \in R^{\nu}.$$

In this case the problem of determining the functions $g^l(x,t)$, $l = 1,2,...,L-1$ is replaced by a problem of identifying the vector $\alpha \in R^{\nu}$.

Thus, we consider the parametrical identification problem (1)-(8) with respect to the finite-dimensional vector $(\alpha, p) \in R^{\nu+r}$. On the one hand, this problem represents a parametrical optimal control problem, and on the other hand, one can consider it as a finite-dimensional optimization problem of specific kind (class).

We obtain formulas providing the components of the gradient of functional (8): $\nabla J(p,g) = (\nabla_{\alpha} J(p,g), \nabla_p J(p,g))$, which make it possible to formulate first-order necessary optimality conditions, as well as to use known efficient first-order numerical methods and provide numerical solutions to the identification problem (1)-(8) [2].

Let (α, p) be some admissible values of the parameters, with respect to which we want to obtain formulas for the gradient of the target functional as well as first order necessary optimality conditions. Suppose that the input data and functions participating in the statement of the considered problem are such that for arbitrary admissible values of the parameters p and possible positions of the discontinuity surface $g(x,t;\alpha) = 0$ from the neighborhood of the values (α, p), the trajectory of the system necessarily hits each discontinuity surface but only once and never slides over it, i.e. there always holds true the condition

$$\left| \left\langle g_x\left(x(\bar{t}_l), \bar{t}_l; \alpha\right), f^l\left(x(\bar{t}_l), p^l\right) \right\rangle + g_t\left(x(\bar{t}_l), \bar{t}_l; \alpha\right) \right| \ge \delta > 0, \quad l = 1,2,...,L-1. \tag{9}$$

Here $\bar{t}_l \in [0,T]$, $l = 1,2,...,L-1$, are moments of time when the trajectory hits the discontinuity surfaces; at that the point and moment of intersection is stable to small perturbations of the parameters. This condition is not of principal value, but the case when it does not hold true necessitates carrying on additional computations for the sections of the trajectory which are on the discontinuity surface.

The following remark is of important value. It is evident that the experiments and results of observations (6) are independent of each other. The same is true for the items of the functional (8). This means that the following formula takes place

$$\nabla J(p,g) = \frac{1}{N} \cdot \sum_{i=1}^{N} \nabla I_i\left(x^i(T; x_0^i, p, g), p, g\right).$$

That is why in order to obtain formulas for $\nabla J(p,g)$ we need to obtain formulas for the gradient with respect to individual items $\nabla I_i\left(x^i(T; x_0^i, p, g), p, g\right)$. To this end we use the formula of the increment of the target functional that is obtained at the expense of the increment of the values of the parameters (α, p).

In the general case for the set of initial conditions and for all $s = 1,2,...,\nu$ we obtain the following formulas for the components of the gradient of the target functional:

$$\begin{aligned} J'_{\alpha_s^l}(p,g) &= \frac{1}{N} \sum_{i=1}^{N} I'_{i\alpha_s^l}\left(x^i(T; x_0^i, p, g), p, g\right), \\ I'_{i\alpha_s^l}\left(x^i(T; x_0^i, p, g), p, g\right) &= \overline{\sigma_l} \cdot \varphi^s\left(x(\bar{t}_l; x_0^i), \bar{t}_l\right), \quad s = 1,2,...,\nu, \ l = 1,2,...,L-1 \end{aligned} \tag{11}$$

where

$$\overline{\sigma_l} = \frac{\psi\left(\overline{t_l}+0\right)\cdot\left[f^{\,l}\left(x\left(\overline{t_l}\right),p^l\right)-f^{\,l+1}\left(x\left(\overline{t_l}\right),p^{l+1}\right)\right]}{\left\langle g_x\left(x\left(\overline{t_l}\right),\overline{t_l};\alpha\right), f^{\,l}\left(x\left(\overline{t_l}\right),p^l\right)\right\rangle + g_t\left(x\left(\overline{t_l}\right),\overline{t_l};\alpha\right)}, \qquad (12)$$

and $\psi(t)=\psi\left(t;x_0^i\right)$ is the solution to the following conjugate system

$$\dot{\psi}^*\left(t;x_0^i\right) = -\psi^*\left(t;x_0^i\right)\cdot\frac{\partial f^{\,l}\left(x(t),p^l\right)}{\partial x},\; t\in\Pi_l\left(x_0^i\right), \qquad (13)$$

$$\psi\left(T;x_0^i\right) = -\frac{\partial I_i\left(x^i\left(T;x_0^i,p,g\right),p,g\right)}{\partial x}, \qquad (14)$$

satisfying the following jump condition at the moment of time when the trajectory of system (1) hits the discontinuity surface

$$\psi\left(\overline{t_l}-0;x_0^i\right) = \psi\left(\overline{t_l}+0;x_0^i\right) - g_x\left(x\left(\overline{t_l}\right),\overline{t_l};\alpha\right)\cdot\overline{\sigma_l}\;,\; l=1,2,...,L-1. \qquad (15)$$

In (13), $\Pi_l\left(x_0^i\right),\; l=1,2,...,L,\; i=1,2,...,N$, designates the period of time during which the trajectory of (1) with initial condition x_0^i and the values of parameters (α,p) is in the zone X^l.

The following theorem holds true.

Theorem (necessary optimality conditions). For the optimality of the vector $\tilde{\alpha}$ in problem (1)-(8), it is necessary that the following relation be satisfied:

$$\left\langle \nabla_\alpha J(p,g),\, \tilde{\alpha}-\alpha\right\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N}\nabla_\alpha I_i\left(x\left(T;x_0^i,p,g\right),p,g\right),\, \tilde{\alpha}-\alpha\right\rangle \ge 0,\; \forall\alpha\in U\left(\tilde{\alpha},\delta\right),$$

where $\nabla_\alpha J(p,g)$ is determined by formulas (11)-(15); $U\left(\tilde{\alpha},\delta\right)$ is δ-neighbourhood of the point $\tilde{\alpha}$.

Generalizing the results of the works [3, 5, 6], using the same scheme for the increment of the target functional as in case of formulas (11), it is not difficult to derive a formula for the gradient of the target functional with respect to the vector of parameters p. This formula is as follows:

$$J'_{p^l}(p,g) = \frac{1}{N}\sum_{i=1}^{N} - \int_{\Pi_l\left(x_0^i\right)} \frac{\partial f\left(x\left(t;x_0^i,p,g\right),p^l\right)}{\partial p^l}\cdot\psi\left(t;x_0^i\right)dt\,. \qquad (16)$$

Here at the interval of time $\Pi_l\left(x_0^i\right),\; l=1,2,...,L,\; i=1,2,...,N$, the vector p takes on the value p^l. It is evident that for each given initial condition x_0^i there is a different period of time when the values of the parameters are constant.

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