The Third International Conference "Problems of Cybernetics and Informatics"
September 6-8, 2010, Baku, Azerbaijan. The plenary paper
www.pci2010.science.az/4/04.pdf

A BRIEF REVIEW ON SOME FRACTIONAL POINT PROCESSES

Enzo Orsingher[1], Federico Polito[2]

Department of Statistics, Probability and Applied Statistics
"Sapienza" University of Rome, Italy
[1]*enzo.orsingher@uniroma1.it*, [2]*federico.polito@uniroma1.it*

In this paper we consider some fractional versions of classical pure birth, pure death and birth and death processes.

The classical non-linear pure birth process is a model of an expanding population where the number of components at time t is a random point process $N(t), t > 0$, with birth rates $\lambda_k, k \geq 1$.

The state probabilities

$$p_k(t) = \Pr\{N(t) = k \mid N(0) = 1\}, \quad k \geq 1, t > 0, \tag{1}$$

satisfy the difference-differential equations

$$\begin{cases} \dfrac{d}{dt} p_k = \lambda_{k-1} p_{k-1} - \lambda_k p_k, & k \geq 1, \\[2mm] p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k \geq 2. \end{cases} \end{cases} \tag{2}$$

In the fractional version of (2), we replace the time derivative with the fractional derivative

$$\frac{d^v}{dt^v} f(t) = \frac{1}{\Gamma(1-v)} \int_0^t \frac{f'(s)}{(t-s)^v} ds, \quad 0 < v < 1, \tag{3}$$

called Caputo or Dzhrbashyan-Caputo derivative.

We have shown, in Orsingher-Polito (2010) that, for the fractional birth process $N^v(t), t > 0$, the solution to

$$\begin{cases} \dfrac{d^v}{dt^v} p_k = \lambda_{k-1} p_{k-1} - \lambda_k p_k, & k \geq 1, \\[2mm] p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k \geq 2. \end{cases} \end{cases} \tag{4}$$

has the form

$$p_k^v(t) = \begin{cases} \Pi_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \dfrac{E_{v,1}(-\lambda_m t^v)}{\Pi_{l=1, l \neq m}(\lambda_l - \lambda_m)}, & k > 1, \\[4mm] E_{v,1}(-\lambda_1 t^v), & k = 1 \end{cases} \tag{5}$$

where

$$E_{v,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(vk+1)}, \quad x \in \Re, v > 0, \tag{6}$$

is the Mittag-Leffler function.

For $v = 1$, we recover, from (5), the classical distribution of the non-linear pure birth process.

We were able to show that, between $N^v(t)$ and $N(t)$, holds the following subordination relation:

$$N^v(t) = N(T_{2v}(t)), \quad t > 0, \tag{7}$$

where $T_{2v}(t), t > 0$, is a random process possessing a law satisfying the fractional diffusion

The Third International Conference "Problems of Cybernetics and Informatics"
September 6-8, 2010, Baku, Azerbaijan. The plenary paper
www.pci2010.science.az/4/04.pdf

equation

$$
\begin{cases}
\dfrac{\partial^{2v}}{\partial t^{v}} u = \dfrac{\partial^{2}}{dx^{2}} u, & 0 < v \le 1,\, t > 0,\, x > 0, \\[2mm]
u(x,0) = \delta(x), & \\[2mm]
u_{t}(x,0) = 0, & for\ 1/2 < v \le 1, \\[2mm]
u_{x}(0,t) = 0.
\end{cases}
\tag{8}
$$

For $v = 1/2$, $T_1(t)$ is a reflecting Brownian motion so that we can write

$$
N^{\frac{1}{2}}(t) = N(|B(t)|), \qquad t > 0.
\tag{9}
$$

The distributional structure (5) is complicated, and some special cases must be considered in order to illustrate what is going on. For the linear case, that is when $\lambda_k = \lambda k, \lambda > 0$, the equations governing the state probabilities become

$$
\begin{cases}
\dfrac{d^{v}}{dt^{v}} p_{k} = \lambda(k-1)p_{k-1} - \lambda k p_{k}, & k \ge 1, \\[2mm]
p_{k}(0) = \begin{cases} 1, k = 1, \\ 0, k \ge 2, \end{cases}
\end{cases}
\tag{10}
$$

and we obtain that

$$
p_{k}^{v}(t) = \sum_{j=1}^{k} \binom{k-1}{j-1}(-1)^{j-1} E_{v,1}(-\lambda j t^{v}), \qquad k \ge 1.
\tag{11}
$$

For $v = 1$, the distribution (11) becomes the geometric distribution

$$
p_{k}^{1}(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}, \qquad k \ge 1.
\tag{12}
$$

Linearity permits us to obtain, in a relatively simple form, the probability generating function $G_{v}(u,t) = E u^{N^{v}(t)}$, $|u| < 1$, $t > 0$, the mean and the variance, which read

$$
\begin{cases}
E N^{v}(t) = E_{v,1}(\lambda t^{v}), \\[2mm]
Var N^{v}(t) = 2 E_{v,1}(2\lambda t^{v}) - E_{v,1}(\lambda t^{v}) - E_{v,1}^{2}(\lambda t^{v}).
\end{cases}
\tag{13}
$$

We can see that, for $0 < v < 1$, the mean number of components of the population increases faster and faster, as v decreases in this interval.

Something similar to what we present here, happens for the pure death process.

We have initially a population consisting of n_0 individuals, subject to non-linear death rates μ_k.

While in the pure birth case (fractional or not), the population expands without restriction, in the pure death case, it is condemned to extinction.

The state probabilities

$$
\hat{p}_{k}^{v}(t) = \Pr\{M^{v}(t) = k \mid M^{v}(0) = n_{0}\}, \qquad 0 \le k \le n_{0},\, t > 0,
\tag{14}
$$

satisfy the fractional equations

$$
\begin{cases}
\dfrac{d^{v}}{dt^{v}} \hat{p}_{k} = \mu_{k+1}\hat{p}_{k+1} - \mu_{k}\hat{p}_{k}, & k \ge 1, \\[2mm]
\hat{p}_{k}(0) = \begin{cases} 1, & k = n_{0}, \\ 0, & 0 \le k < n_{0}. \end{cases}
\end{cases}
\tag{15}
$$

The solution to (15) becomes

$$
\hat{p}_{k}^{v}(t) = E_{v,1}(-\mu_{n_{0}} t^{v}), \qquad k = n_{0},
\tag{16}
$$

15

The Third International Conference "Problems of Cybernetics and Informatics"
September 6-8, 2010, Baku, Azerbaijan. The plenary paper
www.pci2010.science.az/4/04.pdf

$$\hat{p}_k^v(t) = \Pi_{j=k+1}^{n_0} \mu_j \sum {}_{m=k}^{n_0} \frac{E_{v,1}(-\mu_m t^v)}{\Pi_{h=k,h\neq m}^{n_0}(\mu_h - \mu_m)}, \qquad 0 < k < n_0, \tag{17}$$

$$\hat{p}_k^v(t) = 1 - \sum {}_{m=1}^{n_0} \Pi_{h=1,h\neq m}^{n_0}\left(\frac{\mu_h}{\mu_h - \mu_m}\right)E_{v,1}(-\mu t^v), \qquad k = 0, n_0 > 1. \tag{18}$$

In the very special case where $n_0 = 1$, the extinction probability becomes

$$\hat{p}_0^v(t) = 1 - E_{v,1}(-\mu_1 t^v). \tag{19}$$

Also, in this case, linearity $(\mu_k = \mu k)$ simplifies the form of the state probabilities, which become

$$\hat{p}_k^v(t) = \binom{n_0}{k}\sum {}_{r=0}^{n_0-k}\binom{n_0 - k}{r}(-1)^r E_{v,1}(-(k+r)\mu t^v), \qquad 0 \leq k \leq n_0. \tag{20}$$

For $v = 1$, we extract from (20), the classical pure death distribution (Binomial):

$$\hat{p}_k^1(t) = \binom{n_0}{k}e^{-\mu k t}(1 - e^{-\mu t})^{n_0-k}, \qquad t > 0, \ 0 \leq k \leq n_0. \tag{21}$$

The mean value of the fractional linear pure death process $M^v(t), t > 0$, is

$$EM^v(t) = n_0 E_{v,1}(-\mu t^v), \tag{22}$$

and therefore it steadily decreases as t increases.

Another special case of death process is the sublinear fractional pure death process, where the death rates have the form

$$\mu_k = \mu(n_0 + 1 - k), \qquad 0 \leq k \leq n_0. \tag{23}$$

In this case, the death intensities of individuals increases as the population size decreases. We are able to prove that the fractional sublinear death process $\tilde{M}^v(t), t > 0$, has distribution

$$\Pr\{\tilde{M}^v(t) = k \mid \tilde{M}^v(0) = n_0\} = \sum {}_{l=0}^{n_0-k}\binom{n_0 - k}{l}(-1)^l E_{v,1}(-\mu(l+1)t^v), \qquad 1 \leq k \leq n_0 \tag{24}$$

The extinction probability, in the sublinear case, is given by

$$\tilde{p}_0^v(t) = \sum {}_{l=0}^{n_0}\binom{n_0}{l}(-1)^l E_{v,1}(-l\mu t^v). \tag{25}$$

Furthermore, by summing up the probabilities (24), we get that

$$\sum_{k=1}^{n_0} \tilde{p}_k^v(t) = 1 - \sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k E_{v,1}(-k\mu t^v), \tag{26}$$

and thus (24) and (25) form a genuine probability distribution. Curiously enough, the mean value in the sublinear death process, has a rather complicated structure, since it has the following form:

$$E\tilde{M}^v(t) = \sum {}_{k=1}^{n_0}\binom{n_0 + 1}{k+1}(-1)^{k+1} E_{v,1}(-\mu k t^v), \qquad 0 < v \leq 1, \ t > 0. \tag{27}$$

Between the linear birth process $N^v(t), t > 0$, and the sublinear death process $\tilde{M}^v(t), t > 0$, there are some unexpected analogies, for example:

$$\Pr\{N^v(t) = k \mid N^v(0) = 1\} = \Pr\{\tilde{M}^v(t) = n_0 + 1 - k \mid \tilde{M}^v(0) = n_0\}, \qquad 1 \leq k \leq n_0, \tag{28}$$

and

$$\Pr\{N^v(t) > n_0 \mid N^v(0) = 1\} = \Pr\{\tilde{M}^v(t) = 0 \mid \tilde{M}^v(0) = n_0\} \tag{29}$$

The Third International Conference "Problems of Cybernetics and Informatics"
September 6-8, 2010, Baku, Azerbaijan. The plenary paper
www.pci2010.science.az/4/04.pdf

If we assume that the population is simultaneously subject to births and deaths, the analysis of the fractional counterpart of the classical process becomes more complicated; therefore we restrict ourselves only to the fractional linear birth and death process.

The state probabilities, in this case, are subject to the governing equations

$$\frac{d^v}{dt^v} p_k = -(\lambda + \mu) k p_k + \lambda(k-1) p_{k-1} + \mu(k+1) p_{k+1}, \qquad k \geq 0, \tag{30}$$

and

$$p_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1. \end{cases} \tag{31}$$

For $v = 1$, we obtain the well-known distribution

$$\overline{p}_k^1(t) = (\lambda - \mu)^2 e^{-(\lambda-\mu)t} \frac{(\lambda(1 - e^{-(\lambda-\mu)t}))^{k-1}}{(\lambda - \mu e^{-(\lambda-\mu)t})^{k+1}}, \qquad k \geq 1, t < 0, \mu \neq \lambda. \tag{32}$$

For $\mu = \lambda$, we have a very fine distribution which becomes

$$\overline{p}_k^1(t) = \frac{(\lambda t)^{k-1}}{(1 + \lambda t)^{k+1}}, \qquad k \geq 1, t > 0. \tag{33}$$

The extinction probabilities, play a fundamental role, and take the form

$$\overline{p}_0^1(t) = \begin{cases} \dfrac{\lambda t}{1 + \lambda t}, & \lambda = \mu, \\[4mm] \dfrac{\mu - \mu e^{-t(\lambda-\mu)}}{\lambda - \mu e^{-t(\lambda-\mu)}}, & \lambda \neq \mu. \end{cases} \tag{34}$$

What is remarkable about (34) is that they solve Riccati equations. In the case $\lambda = \mu$, this equation has the form

$$\frac{d}{dt} p_0 + 2\lambda p_0 = \lambda + \lambda p_0^2. \tag{35}$$

As must be expected, all these distributions in the fractional case, have a much more complicated structure. The extinction probabilities, in the three cases $\lambda > \mu, \lambda < \mu$, and $\lambda = \mu$, become

$$\overline{p}_0^v(t) = \frac{\mu}{\lambda} - \frac{\lambda - \mu}{\lambda} \sum_{m=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^m E_{v,1}(-m(\lambda - \mu)t^v), \qquad \lambda > \mu, \tag{36}$$

$$\overline{p}_0^v(t) = 1 - \frac{\mu - \lambda}{\lambda} \sum_{m=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^m E_{v,1}(-m(\mu - \lambda)t^v), \qquad \lambda < \mu, \tag{37}$$

$$\overline{p}_0^v(t) = 1 - \int_0^{\infty} e^{-w} E_{v,1}(-\lambda t^v w) dw, \qquad \lambda = \mu. \tag{38}$$

First of all, we note that, for $v = 1$, we get from the above distribution, the classical extinction probabilities of the linear birth and death process. This is almost immediate for $\lambda = \mu$, since $E_{1,1}(-x) = e^{-x}$. Furthermore, we note that, for $t \to \infty$, the extinction happens with probability one for $\mu \geq \lambda$ while, for $\mu < \lambda, \lim_{t \to \infty} \overline{p}_0^v(t) = \mu/\lambda$, as in the classical case.

For the sake of completeness, we give also the state probabilities for $k > 0$. For $\lambda > \mu$ we have that

$$\overline{p}_k^v(t) = \left(\frac{\lambda - \mu}{\lambda}\right) \sum_{l=0}^{\infty} \binom{l+k}{l} \left(\frac{\mu}{\lambda}\right)^l \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} E_{v,1}(-(l+r+1)(\lambda - \mu)t^v), \tag{39}$$

The Third International Conference "Problems of Cybernetics and Informatics"
September 6-8, 2010, Baku, Azerbaijan. The plenary paper
www.pci2010.science.az/4/04.pdf

while, for $\lambda < \mu$,

$$\overline{p}_k^v(t) = \left(\frac{\lambda}{\mu}\right)^{k-1} \left(\frac{\mu-\lambda}{\mu}\right) \sum_{l=0}^{\infty} \binom{l+k}{l} \left(\frac{\lambda}{\mu}\right)^l \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} E_{v,1}(-(l+r+1)(\mu-\lambda)t^v). \quad (40)$$

The case $\lambda = \mu$ is particularly interesting because, in this case, the state probabilities can be expressed in terms of the probability of extinction

$$\Pr\{\overline{N}^v(t) = k\} = \overline{p}_k^v(t) = (-1)^{k-1} \frac{\lambda^{k-1}}{k!} \cdot \frac{d^k}{d\lambda^k} \left[\lambda(1 - \overline{p}_0^v(t))\right] \qquad k \geq 1. \quad (41)$$

Some direct information on the behaviour of the fractional linear birth and death process, can be obtained from the moments. The mean value is

$$\mathrm{E}\overline{N}^v(t) = \begin{cases} E_{v,1}((\lambda-\mu)t^v), & t > 0, \ \lambda \neq \mu, \\ 1, & \lambda = \mu, \end{cases} \quad (42)$$

and the variance, for $\lambda \neq \mu$, reads

$$Var\overline{N}^v(t) = \frac{2\lambda}{\lambda-\mu} E_{v,1}(2(\lambda-\mu)t^v) - \frac{\lambda+\mu}{\lambda-\mu} E_{v,1}((\lambda-\mu)t^v) - E_{v,1}^2((\lambda-\mu)t^v), \quad (43)$$

while, for $\lambda = \mu$, the variance is

$$Var\overline{N}^v(t) = \frac{2\lambda t^v}{\Gamma(v+1)}. \quad (44)$$

From (42) and (43), for $v = 1$, we obtain the classical expressions of the mean and variance of the linear birth and death process. For example, when $v = 1$,

$$Var\overline{N}^1(t) = \begin{cases} \dfrac{\lambda+\mu}{\lambda-\mu} e^{t(\lambda-\mu)}(e^{t(\lambda-\mu)} - 1), & \lambda \neq \mu, \\ 2\lambda t & \lambda = \mu. \end{cases} \quad (45)$$

This shows that, for $\lambda \geq \mu$, the variance increases as t grows to infinity.

As in the previous cases, the subordination relation between the classical linear birth and death process $\overline{N}^1(t)$, $t > 0$, and its fractional version $\overline{N}^v(t)$, $t > 0$, holds and we can write

$$\overline{N}^v(t) = \overline{N}^1(T_{2v}(t)), \quad t > 0, \quad (46)$$

and this has been the main mean of investigation, permitting us to evaluate all the probabilities presented here.

References and comments

For the classical linear birth (death), non-linear birth (death) and birth and death processes, the reader has at his (her) disposal a huge literature. For example in

Bailey, N. T. J., "The Elements of Stochastic Processes with Applications to the Natural Sciences", Wiley, 1964,

many formulae are given (unfortunately without proof in the non-linear cases).
For the fractional calculus, an easy introduction can be found in

Podlubny, I., "Fractional Differential Equations", Academic Press, 1999.

For the fractional Poisson process, the reader can consult the following text:

Cahoy, D. O., "Fractional Poisson processes in terms of alpha-stable densities", Ph.D. thesis, 2007.

The Third International Conference "Problems of Cybernetics and Informatics"
September 6-8, 2010, Baku, Azerbaijan. The plenary paper
www.pci2010.science.az/4/04.pdf

The results of this paper are presented in

Orsingher, E., Polito, F., "Fractional Pure Birth Processes", published online in Bernoulli, 2010.

Orsingher, E., Polito, F., "On a Fractional Linear Birth-Death Process", published online in Bernoulli, 2010.

Orsingher, E., Polito, F., Sakhno, L., "Fractional Non-Linear, Linear and Sublinear Death Processes", submitted, 2010.