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A CONSTELLATION-FREE MOMENTUM EXCHANGE CONCEPT – THE UNIFORM DISTRIBUTION HYPOTHESIS

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1 Foundations

The well-known Maxwell Hypothesis of Briton James Clark Maxwell characterizes in its original formulation the momentum distribution of gas molecules by a centred normal distribution, its variance essentially being determined by temperature T in [K].
It is the merit of Austrian Ludwig Boltzmann to have calculated the momentum distribution for colliding micro-constituents being imposed on a Newtonian dynamics which allowed him to bridge the notions of kinetic energy, represented by the Hamiltonian *H*, with the one of temperature *T* which is not explained in mechanics.

Let the Hamiltonian $H: IU^N := (IR^2)^N \rightarrow IR_+$ of a system of N micro-constituents - in a planar space (according to our computer experiments) - with momenta $u^{(1)}, \ldots, u^{(N)}$ and $IU_i = IR^2$, $i \in IN_N$, as momentum space of the i-th micro – constituent be given by:

(1.1) $$H(u^{(1)}, \ldots, u^{(N)}) := ½ \sum_{j=1}^{N} \langle u^{(j)}, M^{-1}u^{(j)} \rangle ,$$ where

$\langle . , . \rangle$ denotes the standard scalar product on IR^2 and

(1.2) $$M := \begin{pmatrix} m1 & 0 \\ 0 & m2 \end{pmatrix}, \quad m_1, m_2 > 0,$$

a positive definite and symmetric mass-matrix, which is assumed here by reasons of simplicity as a diagonal matrix which also of course is not covered by Newtonian dynamics for the case of $m_1 \neq m_2$. This does not mean any limitation nor for the mathematical treatment nor for computer experimentation, but it allows more specific insights into the topic to be analyzed here; cf. Moeschlin, Grycko (2006b), Chapter 7.

The equilibrium momentum distribution is given in terms of *H* by

(1.3) $$P = \otimes_{j=1}^{N} N(0, \beta H)$$

as the N^{th} power of the normal distribution $N(0, \beta H)$ with $\beta = k_B T$, k_B denoting the Boltzmann constant.
Notice, the normal distribution $N(0, \beta H)$ on the meausurable space (IR^2, B^2) with the Borel σ-field B^2 has in case of $m_1 \neq m_2$ elliptical contours.
(For reasons of simplicity σ-fields will not be mentioned more in the sequel.)

A momentum exchange is – following Boltzmann -- initiated by a collision of (two) micro-constituents in the physical constellation space, where the outcome of such a collision, i.e. the momentum exchange vector, is essentially determined by the momentum exchange direction, i.e. the difference of the positions of the colliding micro-constituents taken as unit vector. Thereby the system of moving and colliding micro-constituents in the physical constellation space might be seen as somewhat like a machine generating randomly these momentum exchanges according - for instance - to some probability law.

The questions to be treated here is: Can the momentum exchange concept of Boltzmann be liberated from the constellations of the molecules in the physical constellation space, can it be liberated from any mechanical dynamics?

In Grycko, Moeschlin (2006a) micro-models of momentum exchange were investigated to explain a postulated momentum distribution on a discrete momentum space (justified by the concept of optimal entropy) being inspired by the harmonic oscillator from quantum physics. The result was that at any exchange of momentum between two momenta the momentum exchange vector had to be chosen according to (the discrete uniform distribution) under those (finite many) elements of the discrete momentum space ensuring the condition of energy conservation.
Indeed, also for the Maxwell-Boltzmann case an affirmative answer can be given; the guideline, although quite different, was delivered by Grycko, Moeschlin (2006a).

2 The set of all possible energy preserving momentum exchange vectors

Depending on the question to be treated it is more sensible not to see the momentum exchange direction as normalized difference of the position-vectors of the micro-constituents but to assume its polar angle φ as given. In this sense define the momentum exchange direction of the micro-constituents i and j with a specified polar angle φ(i,j)=:φ by

(2.1) $e^{(i,j)}_{\varphi} := (\cos\varphi, \sin\varphi)^t,\ \varphi \in [0,2\pi) =: I,\ i,j \in \mathrm{IN}_N,\ i \neq j.$

According to principles of mechanics the momentum exchange vector of the micro-constituents $i,j \in \mathrm{IN}_N$, $i \neq j$, is determined by by the ansatz

(2.2)
$$\underline{u}^{(i)} = u^{(i)} + \xi_\varphi e^{(ij)}_\varphi$$
$$\underline{u}^{(j)} = u^{(j)} - \xi_\varphi e^{(ij)}_\varphi ,$$

where scalar ξ_φ is fixed by the condition of energy conservation, i.e. by

(2.3) $$H_0(u^{(i)}) + H_0(u^{(j)}) = H_0(\underline{u}^{(i)}) + H_0(\underline{u}^{(j)})$$

where $u^{(i)}$, $u^{(j)}$ and $\underline{u}^{(i)}$, $\underline{u}^{(j)}$ denote the momenta of micro-constituents i and j before and after the momentum exchange.

The (energy preserving) momentum exchange vector $a^{(ij)}_\varphi$ of the micro-constituents i and j with momentum exchange direction $e^{(ij)}_\varphi$ is determined by (2.3) as

(2.4) $$a^{(ij)}_\varphi := \xi_\varphi e^{(ij)}_\varphi \quad \text{with}$$
(2.5) $$\xi_\varphi = \langle u^{(i)} - u^{(j)}, M^{-1} e_\varphi \rangle \,/\, \langle e_\varphi, M^{-1} e_\varphi \rangle$$

The *set of all possible energy preserving momentum exchange vectors* for micro-constituents i and j, $i,j \in \mathrm{IN}_N$, $i \neq j$, is introduced observing (2.4), (2.5) as

(2.6) $E^{(ij)} := \{a^{(ij)}_{\varphi} \mid \varphi \in I\}$.

3 Uniform Distribution Hypothesis

Inserting (2.4) into (3.1) it can be shown, that $E^{(ij)}$, $i,j \in IN_N$, $i \neq j$, is an ellipse in IR^2 whose points (y_1,y_2) satisfy the equation (3.1); for details it is referred to Moeschlin, Grycko (2009):

(3.1) $(y_1 - 1/2\ d_1^{(ij)})^2 / m_1 r_{(ij)}^{\ 2} + (y_2 - 1/2\, d_2^{\ (ij)})^2 / m_2\, r_{(ij)}^{\ 2} = 1$ with

(3.2) $r^{(ij)} := ½ \langle d^{(ij)}, M^{-1} (d^{(ij)}) \rangle$ and

(3.3) $(d_1^{(ij)}, d_2^{\ (ij)}) =: d(ij) =: u^{(i)}_{\varphi} - u^{(j)}_{\ \varphi}$.

By the determination of matrix M in (1.2) as a diagonal matrix, the main axes of the ellipses $E^{(ij)}$, $i,j \in IN_N$, $i \neq j$ are parallel to the coordinate axes.

The present research culminates in the *Uniform Distribution Hypothesis:*
:

> *The momentum exchange vectors $a^{(ij)}$ generated by the dynamics of the Boltzmann system, here with mass matrix M according to (1.2), (but of course with also any symmetric and positive definite mass matrix) follow a uniform probability distribution on the ellipses $E^{(ij)}$, $i,j \in IN_N$, $i \neq j$.*

By (3.1) natural parameterizations of $E^{(ij)}$, $i,j \in IN_N$, $i \neq j$ are given by

(3.4) $b^{(ij)} : I = [0,2\pi) \rightarrow E^{(ij)}$ with

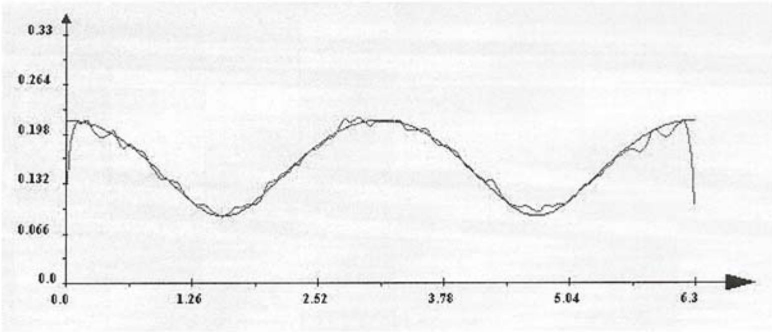
(3.5) $b^{(ij)}(\varphi) := (\sqrt{m1} \cdot r^{(ij)} \cdot \cos\varphi + 1/2\, d_1^{(ij)}),\ \sqrt{m2} \cdot r^{(ij)} \cdot \sin\varphi + 1/2\, d_2^{\ (ij)})$

Preparing a computer experiment the density of a probability measure Pij on $I=[0,2\pi)$ is calculated such that the image of Pij under the mapping $b^{(ij)}$ – call it Uij – are uniform probability distributions on $E^{(ij)}$, $i,j \in IN_N$, $i \neq j$.

These Lebesgue densities of Pij are given $i,j \in IN_N$, $i \neq j$, by the same expression, i.e. by (3.6) not depending on $i,j \in IN_N$, $i \neq j$,

(3.6) $$\frac{(m_1 \sin^2\psi + m_2 \cos^2\psi)^{1/2}}{\int_0^{2\Pi} (m_1 \sin^2\psi + m_2 \cos^2\psi)^{1/2}\ d\psi} \ .$$

For a computer experimental proof the graph of the density (3.6), being unique for all $i,j \in IN_N$, $i \neq j$, is compared with the nonparametric estimate of (3.6), cf. Nadaraya (1989), based on realizations $\psi^k := (b^{(i,j)})^{-1}(\hat{a}^k) \in [0,2\pi)$ with $\hat{a}^k$, $k \in IN$, being realizations of momentum exchange vectors of a computer experiment according to the Boltzmann model being imposed on the Newton dynamics with mass matrix M of (2.1), cf. Moeschlin, Grycko (2006b):



Smooth line: calculated density; peak line: estimated density

The statistical coincidence of the calculated density with its nonparametric estimate is almost total, which confirms the Uniform Distribution Hypothesis.

References

Grycko, E., O. Moeschlin (2006), Entropy principle and momentum exchange on a discrete momentum space . Appl. Comp. Math. 33, No.4, 541-552 (2006)

Moeschlin, O., E. Grycko (2006): Experimental Stochastics in Physics. Springer Publishing House, New York, Berlin, Heidelberg.

Moeschlin, O., E. Grycko (2009): A Probability Law Inherent in the Momentum Exchange Process – the Uniform Distribution Hypothesis. Preprint.

Nadaraya, E.A. (1989): Nonparametric Estimation of Probability Densities and Regression Curves. Dordrecht, Boston, London.