# GREEDY ALGORITHMS WITH GUARANTEE VALUE FOR INTEGER KNAPSACK PROBLEM

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**Introduction**

It is known that *Integer Knapsack Problem* is NP-hard and there is no polinomial time algorithm unless P=NP. Consequently, it is important to design algortihms with guarantee value for NP-hard problems [1, 3, 6]. In recent years, many researches on this subject have been doing [2, 4, 5].

In this paper, one-dimensional *Integer Knapsack Problems* have been studied. Moreover, greedy algorithms have been given to solve these problems, then their guaratee values have been calculated. The concept of complementary problem has been defined for the maximization problem and guaratee values calculated have been improved by this concept.

Integer maximization knapsack problem (IKP) is formulated as

$$R = \max\left\{ \sum_{j\in J} p_j x_j \,\middle|\, \sum_{j\in J} a_j x_j \le b,\ x_j \in Z^+ \cup \{0\},\ j \in J \right\} \tag{1}$$

Coefficients $a_j$, $p_j$, and value $b$ are generally positive integers.

Supposing that $\frac{p_1}{a_1} \ge \frac{p_2}{a_2} \ge ... \ge \frac{p_n}{a_n}$, greedy algorithm , that gives approximate solution for this problem, is given as below:

**<u>Tmax Algorithm</u>**

**A1)** $k = 1$ and $f = 0$;

**A2)** $x_k^G = \left\lfloor \frac{b}{a_k} \right\rfloor$;

**A3)** $f = f + p_k x_k^G$;

**A4)** $b = b - a_k x_k^G$;

**A5)** $k = k + 1$;

**A6)** If $k > n$ then go to Step8;

**A7)** If $a_k \le b$ then go to Step2;

**A8)** $x_k^G = x_{k+1}^G = ... = x_n^G = 0$;

**A9)** Print $X^G, f$ ;

**A10)** END.

**<u>Calculation of Guarantee Value of Tmax Algorithm</u>**

For $f$ -approximate solution found by the algorithm- and $R$ - optimal value- the guaratee value $\Delta = \frac{f}{R}$ is given below. (For $\forall j \in J,\ a_j \le b$ )

Since $R \le \frac{b}{a_1} p_1$ and $f \ge \left\lfloor \frac{b}{a_1} \right\rfloor p_1$ ;

$$\Delta \geq \frac{\left\lfloor \frac{b}{a_1} \right\rfloor p_1}{\frac{b}{a_1} p_1} \geq \frac{\left\lfloor \frac{b}{a_1} \right\rfloor}{\left\lfloor \frac{b}{a_1} \right\rfloor + \left\{ \frac{b}{a_1} \right\}} \geq \frac{\left\lfloor \frac{b}{a_1} \right\rfloor}{\left\lfloor \frac{b}{a_1} \right\rfloor + 1} = \frac{1}{1 + \frac{1}{\left\lfloor \frac{b}{a_1} \right\rfloor}}$$

Let $\left\lfloor \frac{b}{a_1} \right\rfloor = m$, then the result would be $\frac{1}{1+\frac{1}{m}}$; so it is found as

$$\Delta \geq \frac{1}{1+\frac{1}{m}} \quad \Rightarrow \quad \lim_{m \to \infty} \frac{m}{m+1} = 1$$

It means that we obtain better results with the increase of the m value, however, guarantee value would be "**1/2**" in the worst case**.**

$$\left( \frac{m}{m+1} \right) = \alpha \quad \Rightarrow \quad \alpha R \leq f \leq R$$

**Complementary Problem of IKP**

For each $x_i$, we find

$$n_i = \left\lfloor \frac{b}{a_i} \right\rfloor$$

Here, $n_i$ represents how many pieces we can take from $i^{th}$ variable at most. $B = \sum_{i \in J} n_i a_i, \ \ \overline{b} = B - b, \ \ y_i = n_i - x_i$ and the complementary problem occurs as below:

$$\overline{R} = \min \left\{ \sum_{j \in \overline{J}} p_j y_j \ \middle| \ \sum_{j \in \overline{J}} a_j y_j \geq \overline{b}, \ y_j \in Z^+ \cup \{0\}, \ y_j \leq n_j, \ j \in \overline{J} \right\} \tag{2}$$

Notice that it is a bounded integer minimization problem. Without losing generality, let values $a_j$ and $p_j$, $j \in \overline{J}$ be positive integers; besides, $\frac{p_1}{a_1} \leq \frac{p_2}{a_2} \leq \ldots \leq \frac{p_n}{a_n}$

**Tmin Algorithm for Complementary Problem**

**A1)** $k = 1$ and $\overline{f} = 0$;

**A2)** $y_k^G = \left\lceil \frac{\overline{b}}{a_k} \right\rceil$;

**A3)** If $y_k^G > n_k$, then $y_k^G = n_k$, $\overline{f} = \overline{f} + y_k^G p_k$, $\overline{b} = \overline{b} - y_k^G a_k$

else $\overline{f} = \overline{f} + y_k^G p_k$ and go to Step6;

**A4)** $k = k + 1$;

**A5)** If $\overline{b} > 0$ and $k \leq n$ then go to Step2;

**A6)** Print $\overline{Y}^G$, $\overline{f}$ ;

**A7)** END.

**Calculation of Guarantee Value of Tmin Algorithm**

Let $s-1=\max\left\{k \,\middle|\, \sum_{i=1}^{k} n_i a_i < \bar{b}\right\}$, then the result of the algorithm is given as

$$y_i^G = n_i, \qquad i=\overline{1,s-1}$$

$$y_i^G = 0, \qquad i=\overline{s+1,n}$$

$$y_s^G = \left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil, \quad \left(\bar{\bar{b}} = \bar{b} - \sum_{i=1}^{s-1} n_i a_i\right)$$

$$\bar{f} = \sum_{i=1}^{s-1} n_i p_i + \left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil p_s$$

Notice that $\left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil \le n_s$.

If the problem was continuous, the solution would be $\tilde{\bar{R}} = \sum_{i=1}^{s-1} n_i p_i + \frac{\bar{\bar{b}}}{a_s} p_s$,

Furthermore, we know $\tilde{\bar{R}} \le \bar{R} \le \bar{f} \;\Rightarrow\; 2\tilde{\bar{R}} \le 2\bar{R} \le 2\bar{f}$. Now, there are two cases we will observe:

**<u>1st case</u>:** *s>1*

$$a_{\max} = \max\left\{a_j \mid j=1,\dots,n\right\}$$

$$\Delta = \frac{\bar{f}}{\bar{R}} \le \frac{\bar{f}}{\tilde{\bar{R}}} = \frac{\sum_{i=1}^{s-1} n_i p_i + \left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil p_s}{\sum_{i=1}^{s-1} n_i p_i + \frac{\bar{\bar{b}}}{a_s} p_s} = \frac{\left(\sum_{i=1}^{s-1} n_i p_i + \frac{\bar{\bar{b}}}{a_s} p_s\right) + \left(\left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil p_s - \frac{\bar{\bar{b}}}{a_s} p_s\right)}{\sum_{i=1}^{s-1} n_i p_i + \frac{\bar{\bar{b}}}{a_s} p_s}$$

$$= 1 + \frac{\left(\left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil - \frac{\bar{\bar{b}}}{a_s}\right) p_s}{\sum_{i=1}^{s-1} n_i p_i + \frac{\bar{\bar{b}}}{a_s} p_s} \le 1 + \frac{\left(\left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil - \frac{\bar{\bar{b}}}{a_s}\right) p_s}{\sum_{i=1}^{s-1} \frac{b-a_i}{a_i} p_i} \le 1 + \frac{p_s}{(b-a_i)\sum_{i=1}^{s-1} \frac{p_i}{a_i}}$$

$$\le 1 + \frac{p_s}{(b-a_{\max})\frac{p_1}{a_1}\sum_{i=1}^{s-1} 1} \le 1 + \frac{1}{(b-a_{\max})(s-1)}$$

**<u>2nd case:</u>** *s=1*

Notice that $\bar{\bar{b}} = \bar{b}$ and $\sum_{i=1}^{s-1} n_i p_i = 0$

$$\bar{f} = \sum_{i=1}^{s-1} n_i p_i + \left\lceil \frac{\bar{\bar{b}}}{a_s} \right\rceil p_s = \left\lceil \frac{\bar{b}}{a_1} \right\rceil p_1 \qquad \tilde{\bar{R}} = \frac{\bar{b}}{a_1} p_1$$

$$\left\lceil \frac{\bar{b}}{a_1} \right\rceil = \left\lfloor \frac{\bar{b}}{a_1} \right\rfloor + 1 \le \left\lfloor \frac{\bar{b}}{a_1} \right\rfloor + \left\lfloor \frac{\bar{b}}{a_1} \right\rfloor \le 2\left\lfloor \frac{\bar{b}}{a_1} \right\rfloor + 2\left\{ \frac{\bar{b}}{a_1} \right\}$$

$$\bar{f} = \left\lceil \frac{\bar{b}}{a_1} \right\rceil p_1 \le \left( 2\left\lfloor \frac{\bar{b}}{a_1} \right\rfloor + 2\left\{ \frac{\bar{b}}{a_1} \right\} \right) p_1 = 2\tilde{\bar{R}} \le 2\bar{R}$$

If looked at the results, guaratee value will be equavalent to "**2"** in the second case; besides, if denominator equals to 1 in the first case then the guarantee value will be the same. Otherwise, while *s* and $(b - a_{\max})$ are going bigger, it results better.

**Some Theorems**

**Theorem 1:** $R + \bar{R} = P \qquad (P = \sum_{i=1}^{n} n_i p_i)$

**Theorem 2:** $f + \bar{f} < P$

**Improvoment of guarantee Value**

We consider problem (2) for the solution of the problem (1). Let us apply Tmin algorithm for this problem and remark $\tilde{f} = P - \bar{f}$

**Theorem 3:** $R \ge \tilde{f} \ge \begin{cases} \alpha R, & \text{if} \quad \mu < \alpha/(2-\alpha) \\ (2\mu/(1+\mu))R, & \text{if} \quad \mu \ge \alpha/(2-\alpha) \end{cases}$

Here $\mu = \tilde{f}/P$ ,

**Theorem 4:** $\bar{R} \le \bar{f} \le \begin{cases} 2\bar{R}, & \text{if} \quad \lambda > (2\alpha-2)/(\alpha-2) \\ (\alpha\lambda/(\lambda+\alpha-1))\bar{R}, & \text{if} \quad \lambda \le (2\alpha-2)/(\alpha-2) \end{cases}$

Here $\lambda = \bar{f}/P$

## Literature

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