OPTIMIZATION PROBLEM FOR DISCONTINUOUS SYSTEMS

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A class of optimization problems for dynamic processes described by systems of ordinary differential equations, the form of which changes according to the state of the process belonging to one or another subdomain of state space:

$$\dot{x}(t) = f^i(x(t),u(t)), \quad x(t) \in X^i, \quad t \in (0,T], \quad i = 1,2,...,L, \tag{1}$$

where $x(t) \in R^n$ is the vector of process state; $u(t) \in R^r$ is piecewise constant vector-function of control; X^l is subdomain of phase space of every possible state X, i.e. $X^l \subset X \subseteq R^n$, $l = 1,2,...,L$; vector-functions $f_l(.,.), l = 1,2,...,L$ is continuously differentiable on all their arguments, is investigated in the paper. Subdomains of phase space

$$\begin{aligned} &X^l = \{x \in E^n : g^{l-1}(x) > 0, \quad g^l(x) \le 0\}, \quad l = 2,3,...,L-1, \\ &X^1 = \{x \in E^n : g^1(x) \le 0\}, \quad X^L = \{x \in E^n : g^{L-1}(x) > 0\}, \end{aligned} \tag{2}$$

are simply connected and given by their boundaries, determined by continuously differentiable scalar functions $g^l(x), l = 1,2,...,L-1$, at that

$$X^{l_1} \cap X^{l_2} = 0, \quad l_1 \ne l_2, \quad l_1, l_2 = 1,2,...,L, \quad \bigcup_{l=1}^{L} X^l = X \subseteq E^n. \tag{3}$$

Suppose that possible initial states of the process belong to some set $X_0 \subset X$, i.e. $x(0) \in X_0$, at that the distribution function of the initial state $R(X_0)$ is given.

The optimization problem investigated consists in determining $(L-1)$-dimensional vector-function $g(x) = (g^i(x))_{i=1,2,...,L-1}$ and piecewise continuous function $u(t)$, minimizing the functional

$$\bar{J}(u,g) = \int_{X_0} \left[\int_0^T F(x(t),u(t),t)dt + \Phi(x(T)) \right] dR(X_0) / mes(X_0), \tag{4}$$

i.e. the averaged value of the functional

$$J(u,g) = \int_0^T F(x(t),u(t),t)dt + \Phi(x(T)) \tag{5}$$

on initial states from the set X_0, where $F(x(t),u(t),t)$ and $\Phi(x(T))$ are given functions, $x(t) = x(t;x_0)$, $u(t) = u(t;x_0)$.

In this work we propose to parameterize unknown functions $g^l(x), l = 1,2,...,L-1$ with the use of any known finite system of linearly independent continuously differentiable functions $\{\varphi^i(x)\}, i = 1,2,...,\bar{\nu}$:

$$g^l(x) = g^l(x;\alpha) = \sum_{i=1}^{\nu_l} \alpha_i^l \varphi^i(x), \quad l = 1,2,...,L-1, \quad \nu = \sum_{l=1}^{L-1} \nu^l, \quad \bar{\nu} = \max_{1 \le l \le L-1} \nu^l, \tag{6}$$

$\alpha = (\alpha_1^1,...,\alpha_{\nu_1}^1,...,\alpha_{\nu_{L-1}}^{L-1}) \in E^\nu$. In this case the problem of determining optimal functions $g^l(x), l = 1,2,...,L-1$ is replaced by the problem of optimization of the vector α.

Thus, (1)-(6) is a parametrical optimal control problem concerning control $u(t)$ and finite-dimensional vector of parameters α. Necessary optimality conditions for discontinuous systems concerning control $u(t)$, provided that the curves of discontinuity of the system are given, were obtained in the works [1-3]. In this work, necessary first-order optimality conditions concerning the curves of discontinuity of the system and formulas for the components of the gradient of the target functional concerning the arguments α , were obtained.

We will suppose that the admissible controls $u(t)$ and possible positions of the discontinuity curve $g(x,\alpha)=0$, participating in the statement of the problem considered, are such that the trajectory of the system hits each discontinuity surface only once and never "slides" on it, i.e. the following condition takes place:

$$\left\langle g_x\left(x(\bar{t}_l),\alpha\right), f^l\left(x(\bar{t}_l),u(\bar{t}_l)\right)\right\rangle \neq 0, \quad l=1,2,...,L, \tag{7}$$

where $\bar{t}_l \in [0,T]$ is the time when the trajectory hits the switching curve. This condition is not principal, but the case when it does not take place will require carrying out additional computations for the parts of the trajectory lying on the discontinuity curve.

Let's consider a particular case when there are only two subdomains of phase space, i.e. $L=2$ (we will omit the upper index at the function $g^l(x)$ and parameters α_i^l , ν^l). Derive an increment formula for the functional (4) at any initial condition $x(0)=x_0 \in X_0$ concerning the parameters α . Suppose that one of the arguments $(\alpha_i)_{i=1,2,...,\nu}$ of the function $g(x,\alpha)$ obtains an increment, e.g. α_1: $\overline{\alpha_1}=\alpha_1+\Delta\alpha_1$. At that $x(t)$, $\bar{x}(t)=x(t)+\Delta x(t)$ are the trajectories corresponding to them. The increment of the target functional equals:

$$\begin{gathered}\Delta_{\alpha_1} J(u,g;x_0)=\int_{\bar{t}}^{\bar{t}+\Delta\bar{t}} \frac{\partial F(x(t),u(t),t)}{\partial x}\Delta x(t)dt+\int_{\bar{t}+\Delta\bar{t}}^{T} \frac{\partial F(x(t),u(t),t)}{\partial x}\Delta x(t)dt+ \\ +\frac{\partial \Phi(x(T))}{\partial x}\Delta x(T)+\varepsilon_1, \\ \varepsilon_1=\int_{\bar{t}}^{\bar{t}+\Delta\bar{t}} o_1\left(\|\Delta x(t)\|\right)dt+\int_{\bar{t}+\Delta\bar{t}}^{T} o_2\left(\|\Delta x(t)\|\right)dt+o_3\left(\|\Delta x(T)\|\right).\end{gathered} \tag{8}$$

Here $\bar{t}\in[0,T]$ is the time when the trajectory of the system with initial state x_0 hits the switching curve; $\bar{t}+\Delta\bar{t}\in[0,T]$ is the time when the perturbed trajectory $\bar{x}(t)$ hits the switching curve. Below we will suggest that $\Delta\bar{t}>0$. The case when $\Delta\bar{t}<0$ is investigated analogously, at that after proceeding to limit $\Delta\bar{t}\to 0$ at $\Delta\alpha_1\to 0$, the results, as it is simply to check, coincide.

Consider the identities:

$$\begin{gathered}\psi(\bar{t}-0)\Delta x(\bar{t}-0)-\psi(0)\Delta x(0)=\int_0^{\bar{t}-0}\dot{\psi}(t)\Delta x(t)dt+\int_0^{\bar{t}-0}\psi(t)\Delta\dot{x}(t)dt, \\ \psi(T)\Delta x(T)-\psi(\bar{t}+0)\Delta x(\bar{t}+0)=\int_{\bar{t}+0}^{T}\dot{\psi}(t)\Delta x(t)dt+\int_{\bar{t}+0}^{T}\psi(t)\Delta\dot{x}(t)dt,\end{gathered} \tag{9}$$

where $\psi(t)$ is arbitrary everywhere differentiable function, except for the point $t=\bar{t}$. Adding up these equalities, on account of the continuity of the trajectory of the system (1), we obtain:

$$\begin{gathered}\psi(T)\Delta x(T)=\left[\psi(\bar{t}+0)-\psi(\bar{t}-0)\right]\Delta x(\bar{t})+\int_0^{\bar{t}-0}\dot{\psi}(t)\Delta x(t)dt+\int_0^{\bar{t}-0}\psi(t)\Delta\dot{x}(t)dt+ \\ +\int_{\bar{t}+0}^{T}\dot{\psi}(t)\Delta x(t)dt+\int_{\bar{t}+0}^{T}\psi(t)\Delta\dot{x}(t)dt.\end{gathered} \tag{10}$$

On account of the arbitrariness of the function $\psi(t)$ suppose that

$$\psi(T)=-\frac{\partial\Phi(x(T))}{\partial x}. \tag{11}$$

Then, taking into account (10) and (11), from (8) obtain

$$\Delta_{\alpha_1} J(u,g;x_0) = \int_{\bar{t}}^{\bar{t}+\Delta\bar{t}} \frac{\partial F(x(t),u(t),t)}{\partial x}\Delta x(t)dt + \int_{\bar{t}+\Delta\bar{t}}^{T} \frac{\partial F(x(t),u(t),t)}{\partial x}\Delta x(t)dt -$$
$$- \int_0^{\bar{t}-0}\left[\dot{\psi}(t)\Delta x(t)+\psi(t)\Delta\dot{x}(t)\right]dt - \int_{\bar{t}+0}^{T}\left[\dot{\psi}(t)\Delta x(t)+\psi(t)\Delta\dot{x}(t)\right]dt - \tag{12}$$
$$-\left[\psi(\bar{t}+0)-\psi(\bar{t}-0)\right]\Delta x(\bar{t})+\varepsilon_1 .$$

It is clear that the function $\Delta x(t) = \bar{x}(t) - x(t)$ satisfies the following equations

$$\Delta\dot{x}(t) = 0, \quad t\in\left(0,\bar{t}\right], \tag{13a}$$

$$\Delta\dot{x}(t) = \frac{\partial f^1(x(t),u(t))}{\partial x}\Delta x(t) + \left[f^1(x(t),u(t)) - f^2(x(t),u(t))\right] + + o_4\left(\left\|\Delta x(t)\right\|\right),\ t\in\left(\bar{t},\bar{t}+\Delta\bar{t}\right], \tag{13b}$$

$$\Delta\dot{x}(t) = \frac{\partial f^2(x(t),u(t))}{\partial x}\Delta x(t) + o_5\left(\left\|\Delta x(t)\right\|\right), \quad t\in\left(\bar{t}+\Delta\bar{t},T\right]. \tag{13c}$$

Taking (13a-c) into account in (12) and presuming that the function $\psi(t)$ is the solution to the following system of equations

$$\dot{\psi}(t) = \begin{cases} 0, \quad t\in\left[0,\bar{t}\right), \\ \dfrac{\partial F(x(t),u(t),t)}{\partial x} - \psi(t)\dfrac{\partial f^1(x(t),u(t))}{\partial x}, \quad t\in\left[\bar{t},\bar{t}+\Delta\bar{t}\right), \\ \dfrac{\partial F(x(t),u(t),t)}{\partial x} - \psi(t)\dfrac{\partial f^2(x(t),u(t))}{\partial x}, \quad t\in\left[\bar{t}+\Delta\bar{t},T\right), \end{cases} \tag{14}$$

we will obtain

$$\Delta_{\alpha_1} J(u,g;x_0) = \int_{\bar{t}}^{\bar{t}+\Delta\bar{t}} \psi(t)\cdot\left[f^2(x(t),u(t)) - f^1(x(t),u(t))\right]dt + -\left[\psi(\bar{t}+0)-\psi(\bar{t}-0)\right]\Delta x(\bar{t})+\varepsilon_2, \tag{15}$$
$$\varepsilon_2 = \varepsilon_1 + \int_{\bar{t}}^{\bar{t}+\Delta\bar{t}}\psi(t)\cdot o_4\left(\left\|\Delta x(t)\right\|\right)dt + \int_{\bar{t}+\Delta\bar{t}}^{T}\psi(t)\cdot o_5\left(\left\|\Delta x(t)\right\|\right)dt.$$

Applying average theorem to the integral in the right-hand side of (15), for the increment of the target functional we will obtain

$$\Delta_{\alpha_1} J(u,g;x_0) = \psi(\bar{t})\cdot\left[f^2(x(\bar{t}),u(\bar{t})) - f^1(x(\bar{t}),u(\bar{t}))\right]\cdot\Delta\bar{t} + -\left[\psi(\bar{t}+0)-\psi(\bar{t}-0)\right]\cdot\Delta x(\bar{t})+\varepsilon_2. \tag{16}$$

Expressing the increment of time $\Delta\bar{t}$ in (16) through the increment of the parameter α_1 - $\Delta\alpha_1$, having used for this aim the fact that the switching curve function equals zero at the time when the trajectory of the system pass through this curve, we will obtain

$$\left[\sum_{k=1}^{\nu}\alpha_k\cdot\varphi_x^k\left(x(\bar{t})\right)\cdot\left(\dot{x}(\bar{t})+\Delta\dot{x}(\bar{t})\right)+\Delta\alpha_1\cdot\varphi_x^1\left(x(\bar{t})\right)\cdot\left(\dot{x}(\bar{t})+\Delta\dot{x}(\bar{t})\right)\right]\cdot\Delta\bar{t} +$$
$$+\left[\sum_{k=1}^{\nu}\alpha_k\cdot\varphi_x^k\left(x(\bar{t})\right)\right]\cdot\Delta x(\bar{t})+\Delta\alpha_1\cdot\left[\varphi^1\left(x(\bar{t})\right)+\Delta x(\bar{t})\cdot\varphi_x^1\left(x(\bar{t})\right)\right]+o\left(\left|\Delta\bar{t}\right|\right)=0,$$

whence

$$\Delta\bar{t} = -\frac{\varphi^1\left(x(\bar{t})\right)\cdot\Delta\alpha_1+\left\langle g_x\left(x(\bar{t}),\alpha\right),\Delta x(\bar{t})\right\rangle+o\left(\left|\Delta\bar{t}\right|\right)}{\left\langle g_x\left(x(\bar{t}),\alpha\right), f^1\left(x(\bar{t}),u(\bar{t})\right)\right\rangle+\left\langle\varphi_x^1\left(x(\bar{t})\right), f^1\left(x(\bar{t}),u(\bar{t})\right)\right\rangle\cdot\Delta\alpha_1}. \tag{17}$$

Then for the increment of the target functional we will obtain

$$\Delta_{\alpha_1} J(u,g;x_0) = \sigma\cdot\varphi^1\left(x(\bar{t})\right)\cdot\Delta\alpha_1+\sigma\cdot o_6\left(\left|\Delta\bar{t}\right|\right)+\varepsilon_2 + +\left[\psi(\bar{t}+0)-\psi(\bar{t}-0)-g_x\left(x(\bar{t}),\alpha\right)\cdot\sigma\right]\cdot\Delta x(\bar{t}), \tag{18}$$

where

$$\sigma = \frac{\psi(\bar{t}) \cdot \left[f^1\left(x(\bar{t}), u(\bar{t})\right) - f^2\left(x(\bar{t}), u(\bar{t})\right) \right]}{\left\langle g_x\left(x(\bar{t}), \alpha\right), f^1\left(x(\bar{t}), u(\bar{t})\right) \right\rangle + \left\langle \varphi_x^1\left(x(\bar{t})\right), f^1\left(x(\bar{t}), u(\bar{t})\right) \right\rangle \cdot \Delta\alpha_1} . \qquad (19)$$

Suppose that the function $\psi(t)$ satisfies the following condition

$$\psi(\bar{t}+0) - \psi(\bar{t}-0) - g_x(x(\bar{t}), \alpha) \cdot \bar{\sigma} = 0 . \qquad (20)$$

Dividing both parts of expression (18) into $\Delta\alpha_1$, after that proceeding to limit at $\Delta\alpha_1 \to 0$ and taking into account (7), in the general case, for the set of initial states X_0 and for all $k = 1,2,...,\nu$ we will obtain

$$\begin{aligned} &\bar{J}'_{\alpha_k}(u,g) = \int_{X_0} J'_{\alpha_k}(u,g;x_0)\, dR(X_0) / mes(X_0), \quad k = 1,2,...,\nu, \\ &J'_{\alpha_k}(u,g;x_0) = \bar{\sigma}(x_0) \cdot \varphi^k\left(x(\bar{t};x_0)\right), \end{aligned} \qquad (21)$$

where

$$\bar{\sigma} = \frac{\psi(\bar{t}) \cdot \left[f^1\left(x(\bar{t}), u(\bar{t})\right) - f^2\left(x(\bar{t}), u(\bar{t})\right) \right]}{\left\langle g_x\left(x(\bar{t}), \alpha\right), f^1\left(x(\bar{t}), u(\bar{t})\right) \right\rangle} . \qquad (22)$$

Thus, in order to obtain numerical values for the components of the gradient of target functional (4) it is necessary:

1. to solve the initial problem concerning system (1) and to find the time when the trajectory of the system hits the switching curve for every possible initial state $x_0 \in X_h \subset X_0$, where X_h is the set of nodes of grid approximation of the given set of initial states X_0;
2. to solve the initial problem concerning conjugate system (11), (14), (20) and (22);
3. to calculate the gradient of the target functional using formulas (21), (22).

References

1. *Trotsky V.A.* Variational optimization problems of control processes for equations with discontinuous right-hand sides // PMM, 1962, vol.26, issue 2 (in Russian).
2. *Velichenko V.V.* On optimal control problems for equations with discontinuous right-hand sides // Automation and Remote Control, #7, 1966, pp.20-29 (in Russian).
3. *Tadumadze T., Arsenashvili A.* Optimal control of delayed variable structure systems with discontinues initial condition / The First International Conference on Control and Optimization with Industrial Applications. Baku, 2005, p.92.
4. *Ayda-zade K.R., Guliev S.Z.* On a feedback control problem for non-linear systems // Automatic Control and Computer Sciences, #1, Riga, Latvia, 2005, pp.15-23 (in Russian).
5. *Guliev S.Z.* On a problem of parametrical identification of a dynamical system / 6th International ISAAC Congress, Ankara, 2007, p.51.