LINEAR-QUADRATIC PROBLEM OF OPTIMAL CONTROL FOR VARIABLE STRUCTURE SYSTEMS

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In the study is considered linear-quadratic problem of the optimal control for one system with variable structure.

Let us consider the problem of minimization of quadratic functional:

$$J(u,v) = \frac{1}{2}x'(t_1)N_1x(t_1) + \frac{1}{2}\int_{t_0}^{t_1}\left[x'(t)N_2(t)x(t) + u'(t)N_3(t)u(t)\right]dt + \tag{1}$$

$$+\frac{1}{2}y'(t_2)M_1y(t_2) + \frac{1}{2}\int_{t_1}^{t_2}\left[y'(t)M_2(t)y(t) + v'(t)M_3(t)v(t)\right]dt$$

at the restrictions

$$\begin{aligned} &u(t) \in U \subset R^r,\ t \in [t_0, t_1], \\ &v(t) \in V \subset R^q\ ,\ t \in [t_1, t_2], \end{aligned} \tag{2}$$

$$\begin{aligned} &\dot{x} = A_1(t)x + B_1(t)u\ ,\ t \in [t_0, t_1], \\ &x(t_0) = x_0\ , \end{aligned} \tag{3}$$

$$\begin{aligned} &\dot{y} = A_2(t)y + B_2(t)v\ ,\ t \in [t_1, t_2], \\ &y(t_1) = Cx(t_1)\ . \end{aligned} \tag{4}$$

Here it is assumed that $x(t)$ $(y(t))$ $n\ (m)$-dimensional vector of phase variables, t_0, t_1, t_2 $(t_0 < t_1 < t_2)$ are given, N_1, M_1 are given constant matrixes of the corresponding dimension, $A_i(t), B_i(t), i = 1,2,\ N_i(t), M_i(t), i = 2,3$ are given continuous matrix functions of the corresponding dimension, C is given constant matrix, x_0 is given constant vector, $u(t)$ $(v(t))$ is $r\ (q)$-dimensional sectionally continuous vector of the control actions, U, V are nonempty, bounded and open sets.

Let name pair $(u(t), v(t))$ which satisfies all the foregoing conditions as permissible control, and corresponding process name $(u(t), v(t), x(t), y(t))$ permissible process.

Considering $(u(t), v(t), x(t), y(t))$ fixed permissible process let us introduce functions of Hamilton-Pontryagin:

$$H_1(t, x, u, \psi_1) = -\frac{1}{2}\left[x'N_2(t)x + u'N_3(t)u\right] + \psi_1'\left[A_1(t)x + B_1(t)u\right],$$

$$H_2(t, y, v, \psi_2) = -\frac{1}{2}\left[y'M_2(t)y + v'M_3(t)v\right] + \psi_2\left[A_2(t)y + B_2(t)v\right].$$

Here $\psi_i = \psi_i(t),\quad i = 1,2$ are vector-functions of the corresponding dimensions, which are also solutions of the following system of linear-nonhomogeneous differential equations:

$$\begin{aligned} &\dot{\psi}_1 = -A_1'(t)\psi_1 + N_2(t)x(t),\quad t \in [t_0, t_1], \\ &\psi_1(t_1) = -N_2x(t_1) + C\psi_2(t_1), \end{aligned} \tag{5}$$

$$\begin{aligned} &\dot{\psi}_2 = -A_2'(t)\psi_2 + M_2(t)y(t),\quad t \in [t_0, t_1], \\ &\psi_2(t_2) = -M_1y(t_2). \end{aligned} \tag{6}$$

Suppose that

$$N_1 \ge 0,\ N_2 \ge 0,\ N_3(t) > 0,$$
$$M_1 \ge 0,\ M_2 \ge 0,\ M_3(t) > 0. \tag{7}$$

Taking in account suppositions (7) it is proved that (for example, using scheme discussed in study [1]) for the optimality of the permissible control $(u(t), v(t))$ is necessary and sufficient fulfilment of the conditions

$$\frac{\partial H_1(t, x(t), u(t), \psi_1(t))}{\partial u} = 0, \quad t \in [t_0, t_1],$$
$$\frac{\partial H_2(t, y(t), v(t), \psi_2(t))}{\partial v} = 0, \quad t \in [t_1, t_2]. \tag{8}$$

Considering expression for H_i, $i = 1,2$ relations (8) are written as:

$$B_1'(t)\psi_1(t) - N_3(t)u(t) = 0,$$
$$B_2'(t)\psi_2(t) - M_3(t)v(t) = 0.$$

Relations (8) result that control $(u(t), v(t))$ is defined by the following formulas:

$$\left.\begin{aligned} u(t) &= N_3^{-1}(t)B_1'(t)\psi_1(t) \\ v(t) &= M_3^{-1}(t)B_2'(t)\psi_2(t) \end{aligned}\right\} \tag{9}$$

Substituting expression (9) in (3) we obtain:

$$\dot{x}(t) = A_1(t)x(t) + B_1(t)N_3^{-1}(t)B_1'(t)\psi_1(t),$$
$$x(t_0) = x_0, \tag{10}$$
$$\dot{y}(t) = A_2(t)y(t) + B_2(t)M_3^{-1}(t)B_2'(t)\psi_2(t),$$
$$y(t_1) = Cx(t_1),$$

$$\dot{\psi}_1 = -A_1'(t)\psi_1 + N_2(t)x(t),$$
$$\psi_1(t_1) = -N_2 x(t_1) + C\psi_2(t_1), \tag{11}$$
$$\dot{\psi}_2 = -A_2'(t)\psi_2 + M_2(t)y(t),$$
$$\psi_2(t_2) = -M_1 y(t_2).$$

Supposing that

$$D_1(t) = B_1(t)N_3^{-1}(t)B_1'(t),$$
$$D_2(t) = B_2(t)M_3^{-1}(t)B_2'(t)$$

the problem (10) can be written as:

$$\dot{x}(t) = A_1(t)x(t) + D_1(t)\psi_1(t),$$
$$\dot{y}(t) = A_2(t)y(t) + D_2(t)\psi_2(t), \tag{12}$$
$$x(t_0) = x_0, \quad y(t) = Cx(t).$$

Then we will seek $\psi_i = \psi_i(t), \quad i = 1,2$ in the following form:

$$\psi_1(t) = -P_1(t)x(t), \tag{13}$$
$$\psi_2(t) = -P_2(t)y(t), \tag{14}$$

where $P_i(t), i = 1,2$ are still unknown symmetrical matrixes.

Differentiating both parts of (13) and (14) with respect to t we obtain:

$$\dot{\psi}_1(t) = -\dot{P}_1(t)x(t) - P_1(t)\dot{x}(t), \tag{15}$$
$$\dot{\psi}_2(t) = -\dot{P}_2(t)y(t) - P_2(t)\dot{y}(t). \tag{16}$$

Considering (11), (12) following relations can be obtained from (15),(16):

$$-A_1'(t)\psi_1(t)+N_2(t)x(t)=-\dot{P}(t)x(t)-P_1(t)\left[A_1(t)x(t)+D_1(t)\psi_1(t)\right],$$

$$-A_2'(t)\psi_2(t)-M_2(t)y(t)=-\dot{P}_2(t)y(t)-P_2(t)\left[A_2(t)y(t)+D_2(t)\psi_2(t)\right],$$

$$\left[P_1(t)D_1(t)-A_1'(t)\right]\psi_1(t)=-\dot{P}_1(t)x(t)-P_1(t)A_1(t)x(t)-N_2(t)x(t),$$

$$\left[P_2(t)D_2(t)-A_2'(t)\right]\psi_2(t)=-\dot{P}_2(t)x(t)-P_2(t)A_2(t)y(t)-M_2(t)y(t),$$

$$\begin{cases}\left[A_1'(t)-P_1(t)D_1(t)\right]P_1(t)x(t)=-\left[\dot{P}_1(t)+P_1(t)A_1(t)+N_2(t)\right]x(t),\\ \left[A_2'(t)-P_2(t)D_2(t)\right]P_2(t)y(t)=-\left[\dot{P}_2(t)+P_2(t)A_2(t)+M_2(t)\right]y(t),\end{cases}$$

$$\begin{cases}\left[A_1'(t)P_1(t)-P_1(t)D_1(t)P_1(t)+\dot{P}_1(t)+P_1(t)A_1(t)+N_2(t)\right]x(t)=0,\\ \left[A_2'(t)P_2(t)-P_2(t)D_2(t)P_2(t)+\dot{P}_2(t)+P_2(t)A_2(t)+M_2(t)\right]y(t)=0.\end{cases}$$

Last expression will be fulfilled for all $x(t)$ and $y(t)$ if

$$\dot{P}_1(t)=-A_1'(t)P_1(t)-P_1(t)A_1(t)-N_2(t)+P_1(t)D_1(t)P_1(t), \quad (17)$$

$$\dot{P}_2(t)=-A_2'(t)P_2(t)-P_2(t)A_2(t)-M_2(t)+P_2(t)D_2(t)P_2(t). \quad (18)$$

Let us find initial conditions. It is clear that

$$\psi_1(t_1)=-P_1(t_1)x(t_1),$$

$$\psi_2(t_2)=-P_2(t_2)y(t_2),$$

Therefore

$$\begin{cases}-N_1x(t_1)+C\psi_2(t_1)=-P_1(t_1)x(t_1),\\ -N_2y(t_2)=-P_2(t_2)y(t_2),\end{cases}$$

$$-N_1x(t_1)-CP_2(t_1)y(t_1)=-P_1(t_1)x(t_1),$$

$$-N_1x(t_1)-CP_2(t_1)Cx(t_1)=-P_1(t_1)x(t_1),$$

This implies that

$$\begin{cases}P_1(t_1)=N_1+CP_2(t_1)C,\\ P_2(t_2)=N_2.\end{cases} \quad (19)$$

Let us name equations (17), (18) with initial conditions (19) system of matrix differential equations Rickaty in the observed problem. If problems (17)-(20) are solved then control $\left(u(t),v(t)\right)$ can be expressed by formula:

$$u(t)=-N_3^{-1}(t)B_1'(t)P_1(t)x(t),$$

$$v(t)=-M_3^{-1}(t)B_2'(t)P_2(t)y(t).$$

Literature

1. E.B.Lee, L.Marcus. Foundation of optimal control theory. New-York, London, Sydney. (1967), 576 p.
2. Afanasiev V.N., Kolmanovsky V.B., Nosov V.R. Mathematical theory building systems automatic control, M.: High school, 1998, 574 p. (in Russian)