STOCHASTIC INTEGRAL REPRESENTATION OF POISSON FUNCTIONALS

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0. In the theory of random processes special place take the so-called martingale representation theorems which, for example, implies the representation of the Wiener and Poisson functionals in the form of stochastic integrals. In the 80th of the past century, it turned out (see Harison and Pliska (1981)) that the martingale representation theorems (along with the Girsanov's measure change theorem) play an important role in the modern financial mathematics. In particular, using the integrand of the stochastic integral appearing in the integral representation, one can construct hedging strategies in the European options of different type.

According to the well-known result obtained by Clark (1970), if F is a $\Im_T$-measurable random variable with $EF^2 < \infty$, then there exists the adapted process $\varphi_t(\omega) \in L_2([0,T] \times \Omega)$, such that the integral representation

$$F = EF + \int_0^T \varphi_t(\omega) dw_t \quad (P\text{-a.s.})$$

holds. However, this result says nothing on finding the process $\varphi_t(\omega)$ explicitly. In this direction we are familiar with one sufficiently general result, the so-called Ocone-Clark's formula by which for the Wiener functionals: $\varphi_t(\omega) = E[D_t^w F \,|\, \Im_t^w](\omega)$, where $D_t^w F$ is the stochastic derivative (so-called the Malliavin's derivative) of the functional F. Another distinct method of finding an integrand $\varphi_t(\omega)$ belongs to Shyryaev, Yor (2003), when the functional ξ is of "maximal" type. With the functional they linked the associated Lewy's martingale and used the generalized Ito's formula. According to the this result, if $S_t = \max_{u \le t} w_u$, then the following integral representation is holds:

$$S_T = ES_T + 2\int_0^T [1 - \Phi(\frac{S_t - w_t}{\sqrt{T-t}})] dw_t \,,$$

where $\Phi(\cdot)$ is the standard normal distribution function.

One should note that application of the Ocone-Clark formula needs as a rule, on the one hand, essential efforts, and, on the other hand, in the cases if the functional F has no the stochastic derivative, its application is impossible. Our approach within the classical Ito's calucus allows one to construct $\varphi_t(\omega)$ explicitly by using both the standard L_2 theory and the theories of weighted Sobolev spaces, if the functional F has no stochastic derivative (see Jaoshvili, Purtukhia (2005)). It is known that the events indicator has, in general, no stochastic derivative: more exactly, the indicator function of $A \in \Im_T^w$ belongs to $D_{2,1}^w$ (where $D_{2,1}^w$ denotes the space of square integrable Wiener functionals having the first order stochastic derivative) if and only if $P(A)$ equal to zero or one. Consequently, one cannot apply the Ocone-Clark formula for the indicator $I_{\{w_T > 0\}}$ ($P(w_T > 0) = 1/2$), whereas our approach allows one to write the following representation:

$$I_{\{w_T>0\}} = EI_{\{w_T>0\}} + \int_0^T \Phi_{0,T-t}(w_t)dw_t \ ,$$

where $\Phi_{0,T-t}(\cdot)$ is the function of a normal distribution with parameters 0 and $T-t$.

The subsequent generalization of the Ocone-Clark formula to the so-called normal martingales (the martingale is said to be normal, if $\langle M,M\rangle_t = t$) is due to Ma, Protter, Martin (1998). According to this formula, if $F \in D_{2,1}^M$, then the Ocone-Haussmann-Clark's representation

$$F = E(F) + \int_0^T {}^p(D_t^M F)dM_t$$

is valid; here $D_{2,1}^M$ denotes the space of quadratically integrable functionals having the first order stochastic derivative, and ${}^p(D_t^M F)$ is the predictable projection of the stochastic derivative $D_t^M F$ of the functional F. But, in this case (exactly, when the quadratic variation $[M,M]$ is not deterministic), unlike the Wiener's one (see, example from Ma, Protter, Martin (1998)), it is impossible to define in a generally adopted manner an operator of stochastic differentiation to obtain the structure of Sobolev spaces, which allows one to construct explicitly the stochastic derivative operator in many cases. In reality, the example from Ma, Protter, Martin (1998) shows that two definitions, Sobolev space and chaos expansion, are compatible if and only if $[M,M]$ is deterministic. Therefore in the martingale case the space $D_{p,1}^M$ ($1 < p < 2$) cannot be defined in the usual way -- i.e., by closing the class of smooth functionals with respect to the corresponding norm. In work of Purtukhia (2003) the space $D_{p,1}^M$ ($1 < p < 2$) is proposed for a class of normal martingales and the integral representation formula of Ocone-Haussmann- Clark is established for functionals from this space.

Since, the Compensated Poisson process belongs to a class of normal martingales -- $\langle M,M\rangle_t = t$, but its quadratic variation is not deterministic -- $[M,M]_t = M_t + t$, consequently, the Ocone-Haussmann-Clark's formula makes it impossible to construct explicitly the operator of the stochastic derivative of the functionals of the Compensated Poisson process, saying nothing on the construction of its predictable projection. Our approach within the framework of nonanticipative stochastic calculus of semimartingales allows one to construct explicitly the expression for the integrand of the stochastic integral in the theorem of martingale representation for square integrable functionals of the Compensated Poisson process, and to derive the formula allowing one to construct explicitly predictable projections of their stochastic derivatives (see Jaoshvili, Purtukhia (2007)).

1. Let $(\Omega, \Im, \mathrm{P}, (\Im_t)_{0\le t\le\infty})$ be a filtered probability space satisfying the usual conditions. Assume that the standard Poisson process N_t is given on it $\mathrm{P}(N_t = n) = e^{-t}t^n / n!$, $n = 0,1,2,...$ and that $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_T$. Denote $M_t := N_t - t$.

Let $Z^+ = \{0,1,2,...\}$ and $P = \{P_1, P_2, P_3,...\}$ -- be the Poisson distribution: $P_x = e^{-T}T^x / x!$, $x = 0,1,2,...$. Let us denote $\Delta f(x) = f(x) - f(x-1)$ ($f(x) := 0$, if $x < 0$) and define the Poisson-Sharle's polynomials: $\Pi_n(x) = [(-1)^n \Delta^n P_x]/P_x$, $n \ge 1$; $\Pi_0 = 1$.

It is wellknown from the course of Functional Analysis that the sequence $\{\pi_n(x)\}_{n\ge 0}$ ($\pi_n(x) = \Pi_n(x)/c_n$) is a basis in the space $L_2(Z^+)$ ($L_2(Z^+) = \{f : \sum_{x=0}^{\infty} f^2(x) < \infty\}$).

Let $\rho(x,T) := e^{-T}T^x / x!$ and denote by $L_2^T := L_2(Z^+; \rho(x,T))$ the functional space on Z^+ with the finite norm $\|g\|_{2,T} := \|g\rho^{1/2}(T)\|_{L_2}$.

Proposition 1.1. The space L_2^T is a Banach space with basis $\{ x^n \rho(x,T),\ n \geq 1\}$.

Proposition 1.2. If $F(\cdot - T) \in L_2^T$, then the stochastic integral $\int_0^T E[F(M_T) \mid \Im_{t-}]dM_t$ is well defined.

Let us denote $\nabla_x f(x) := f(x+1) - f(x)$, ($\nabla_x F(M_T) := \nabla_x F(x)\big|_{x=M_T}$).

Theorem 1.1. If $F(\cdot) \in L_2^T$ and for some number $0 < \alpha < 1$: $\nabla_x F(\cdot - T) \in L_2^{T/\alpha}$, then the stochastic integral below is well defined and the following representation is valid:

$$F(M_T) = E[F(M_T)] + \int_0^T E[\nabla_x F(M_T) \mid \Im_{t-}]dM_t \qquad (P\text{-a.s.}). \tag{1.1}$$

Theorem 1.2. In the conditions of the Theorem 1.1 the following relation holds:

$$^p[D_t^M F(M_T)] = E[\nabla_x F(M_T) \mid \Im_{t-}] \qquad (dP \otimes ds\text{-a.s.}), \tag{1.2}$$

where $^p[D_t^M F(M_T)]$ denotes the predictable projection of the stochastic derivative (with respect to the Compensated Poisson process) $D_t^M F(M_T)$ of the functional $F(M_T)$.

Example 2.1. The random variable M_T^2 has the following stochastic integral representation:

$$M_T^2 = E[M_T^2] + \int_0^T (1 + 2M_{t-})dM_t \qquad (P\text{-a.s.})$$

(note that in the Wiener process cases the Ocone-Clark's formula gives us that:

$$w_T^2 = E[w_T^2] + \int_0^T 2w_t dw_t \qquad (P\text{-a.s.})).$$

2. Fix now $0 \leq S \leq T \leq \infty$ and let us denote

$$\rho(x,y,S,T) = \frac{S^x}{x!} \cdot e^{-S} \cdot \frac{T^{y-x}}{(y-x)!} \cdot e^{-(T-S)}.$$

Denote by $L_2^{S,T} := L_2(Z^+ \times Z^+; \rho(x,y,S,T))$ the functional space on $Z^+ \times Z^+$ with the finite norm $\|g\|_{2,S,T} := \|g\rho^{1/2}(S,T)\|_{L_2}$.

Proposition 2.1. The space $L_2^{S,T}$ is a Banach space with basis $\{ x^n y^m \rho(x,y,S,T),\ n,m \geq 1\}$.

For any function of two variables $g(\cdot,\cdot)$ introduce the designation:

$$\nabla^2 g(x,y) = g(x+1, y+1) - g(x,y).$$

It is not difficult to see that

$$\nabla_x[\nabla_y g(x,y)] = \nabla_y[\nabla_x g(x,y)]$$

and

$$\nabla^2 g(x,y) = \nabla_x[\nabla_y g(x,y)] + \nabla_x g(x,y) + \nabla_y g(x,y).$$

Let us denote

$$\nabla_t^2 g(M_S, M_T) = \nabla_x[\nabla_y g(M_S, M_T)] I_{[0,S]}(t) I_{[0,T]}(t) +$$
$$+\nabla_x g(M_S, M_T) I_{[0,S]}(t) + \nabla_y g(M_S, M_T) I_{[0,T]}(t).$$

It is clear that for any function of two variables $g(\cdot,\cdot)$ there exists the function $h(\cdot,\cdot)$, such that $g(x,y) = h(x, y-x)$. Let us denote $\overline{g}(x,y) = g(x-S, y-x-(T-S))$.

Theorem 2.1. If $F(\cdot,\cdot) \in L_2^{S,T}$ and for some number $0 < \alpha < 1$: $\nabla_x[\nabla_y \overline{F}(\cdot,\cdot)] \in L_2^{S/\alpha, T/\alpha}$, then the stochastic integral below is well defined and the following stochastic integral representation is valid:

$$F(M_S, M_T) = E[F(M_S, M_T)] + \int_0^T E[\nabla_t^2 F(M_S, M_T) \mid \Im_{t-}] dM_t \quad (P\text{-a.s.}). \tag{2.1}$$

Theorem 2.2. In the conditions of the Theorem 2.1 the following relation holds:

$$^p[D_t^M F(M_S, M_T)] = E[\nabla_t^2 F(M_S, M_T) \mid \Im_{t-}] \quad (dP \otimes ds\text{-a.s.}), \tag{2.2}$$

where $^p[D_t^M F(M_S, M_T)]$ denotes the predictable projection of the stochastic derivative (with respect to the Compensated Poisson process) $D_t^M F(M_S, M_T)$ of the functional $F(M_S, M_T)$.

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