HYPERBOLIC BROWNIAN MOTION AND OTHER RANDOM MOTIONS ON HYPERBOLIC SPACES

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1. Hyperbolic spaces

The best known hyperbolic space is the Poincaré half-plane consisting of the set of points $H_2^+ = \{(x,y): y>0\}$ endowed with the metric

$$ds^2 = \frac{dx^2+dy^2}{y^2}.$$

In $H_2^+ = \{(x,y): y>0\}$ the position of points can be defined by means of cartesian coordinates (x,y) or by hyperbolic coordinates (η,α) which are connected by means of the formulas

$$x = \frac{\sinh\eta\cos\alpha}{\cosh\eta - \sinh\eta\sin\alpha} \qquad \eta>0,$$
$$y = \frac{1}{\cosh\eta - \sinh\eta\sin\alpha} \qquad -\frac{\pi}{2}<\alpha<\frac{\pi}{2}.$$

The geodesic lines in $H_2^+ = \{(x,y): y>0\}$ are represented by half-circles with center on y=0 or vertical half-lines, the hyperbolic distance between two arbitrary points (x_1,y_1) and (x_2,y_2) of $H_2^+ = \{(x,y): y>0\}$ is given by

$$\cosh\eta = \frac{(x_1-x_2)^2 + y_1^2 + +y_2^2}{2y_1y_2}.$$

For a right hyperbolic triangle the following Pythagorean theorem holds

$$\cosh\eta = \cosh\eta_1 \cosh\eta_2 .$$

Another important hyperbolic space is the Poincaré disk D={(u,v): $u^2+v^2<1$ } and points $(x,y)\in H_2^+ = \{(x,y): y>0\}$ are mapped onto D by means of the conformal transformation

$$w = \frac{iz+1}{z+i}.$$

2. Hyperbolic Brownian motion in H_2^+

Hyperbolic Brownian motion in H_2^+ is a diffusion with generator

$$\frac{1}{2} y^2 \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\}.$$

The transition function $p_H(x,y,t)$ is the solution to the Cauchy problem

$$\frac{\partial p_H}{\partial t} = \frac{1}{2} y^2 \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} p_H$$
$$p_H(x,y,0) = \delta(x)\delta(y-1).$$

The previous problem in hyperbolic coordinates reads

$$\frac{\partial p_H}{\partial t} = \frac{1}{2} \left\{ \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} (\sinh \eta \frac{\partial}{\partial \eta}) + \frac{1}{\sinh^2 \eta} \frac{\partial^2}{\partial \alpha^2} \right\} p_H,$$

from which we extract the Cauchy problem for the hyperbolic distance

$$\frac{\partial p_H}{\partial t} = \frac{1}{2} \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} (\sinh \eta \frac{\partial}{\partial \eta}) p_H, \qquad \eta, t > 0$$
$$p_H(\eta, 0) = \delta(\eta).$$

The solution to the previous problem (after the time change t′=t/2) is

$$p_H(\eta,t) = \frac{\exp\left\{-\frac{t}{4}\right\}}{\sqrt{\pi}(\sqrt{2t})^3} \int_\eta^\infty \frac{\phi \exp\left\{-\frac{\phi^2}{4t}\right\}}{\sqrt{\cosh \eta - \cosh \phi}} d\phi.$$

From the form of the generator $\frac{1}{2} y^2 \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\}$ we can see that the coordinates (X,Y) are solutions to the stochastic differential system

dX=YdB_1, X(0)=0,
dY=YdB_2, Y(0)=1.

The solution to the previous problem is given by

$$X(t) = \int_0^t e^{B_2(s)-\frac{s}{2}} dB_1(s)$$

$$Y(t) = e^{B_2(s)-\frac{t}{2}}$$

where B_1 , B_2 are independent Brownian motions.

3. Hyperbolic Brownian motion in D

The law of Brownian in the disk D is solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2^2}(1-x^2-y^2)^2 \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} p$$

in cartesian coordinates,

$$\frac{\partial p}{\partial t} = \frac{1}{2^2}(1-r^2)^2 \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right\} p$$

in polar coordinates,

$$\frac{\partial p}{\partial t} = \frac{1}{2} \left\{ \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} \left(\sinh \eta \frac{\partial}{\partial \eta} \right) + \frac{1}{\sinh^2 \eta} \frac{\partial^2}{\partial \alpha^2} \right\} p$$

in hyperbolic coordinates. The initial conditions must be written accordingly.
A basic fact about the above equations is that

$$G^\nu(r,\theta;\phi) = \left(\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} \right)^\nu$$

is the eigenfunction corresponding to the hyperbolic Laplacian, that is

$$\frac{1}{2^2}(1-r^2)^2 \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right\} G^\nu(r,\theta;\phi) = \nu(\nu-1)G^\nu(r,\theta;\phi).$$

The Poisson kernel $G^\nu(r,\theta;\phi)$ can also be given in hyperbolic coordinates η, φ as

$$\left(\frac{1}{\cosh \eta + \sinh \eta \cos \phi} \right)^\nu$$

and possesses the property that

$$\int_0^{2\pi} \left(\frac{1}{\cosh\eta + \sinh\eta\cos\phi} \right)^{\nu} d\phi = P_{-\nu}(\cosh r).$$

4. Random motions at finite velocity

The hyperbolic Brownian motion hardly reflects the underlying structure of the hyperbolic space where it develops. Motions at finite velocity on geodesic lines have been introduced and analyzed in Orsingher and De Gregorio (2007) and Cammarota and Orsingher (2008).

In particular, in the last paper a motion on half-circle geodesics at hyperbolic constant velocity c is studied when deviations on orthogonal lines are assumed to occur at Poisson times.

The explicit form of the mean distance of the randomly moving point is obtained and reads

$$E\cosh\eta(t) = e^{-\frac{\lambda t}{2}} \left\{ \cosh\frac{t\sqrt{\lambda^2+4c^2}}{2} + \frac{\lambda}{\sqrt{\lambda^2+4c^2}} \sinh\frac{t\sqrt{\lambda^2+4c^2}}{2} \right\}$$

where λ is the rate of the Poisson process.

Many other properties of this finite-velocity motion are derived and its version adapted on the sphere are also considered.

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