ON PROBLEM OF EXPRESSIVENESS OF DATABASE QUERIES

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In relational model of databases the state of a database is understood as a finite set of relations between elements. Names of relations and its arities are fixed and refer to *as the circuit of a database*. The separate information stored in the relations of the given circuit, refers to *as a state of a database*. Though relational databases have been thought up for finite data sets, it is frequently convenient to assume that there is an infinite *domain* – for example, the integer or rational numbers – so elements of the data get out of this domain. Functions and relations determined on the entire domain (for example, $<$ and $+$) can be used also at querying. For example, if as a language of queries we use the language of logic of predicates of the first order then queries can use both relations of a database and the relation of the domain, thus variables change on the entire domain.

The signature of a relational structure *L* is non-empty set with the mapping assigning to each relational symbol in *L* the relation of the same arity over this set. Let *M* be an infinite structure of signature *L*. Here we consider the ordered structures. This means that *L* includes a binary relational symbol $<$ of which the interpretation in *M* satisfies to axioms of the linear order. We fix the circuit of database SC and enter the following notations:

$$L_0 = \{<\},\ L' = L_0 \cup SC,\ L'' = L \cup SC.$$

A query of a database can be formally determined as a mapping which is accepted by the state of a database and it makes a new relation of fixed arity over *M*. We consider two languages for querying. Queries of the first language are formulas of signature L' – we name them by *limited*. Queries of the second language are formulas of signature L'' – we name them by *expanded*.

Definition 1. *k*-ary query Θ is *locally generic over finite states* if $\bar{a} \in \Theta$ if and only if $\varphi(\bar{a}) \in \Theta(\varphi(s))$ for any partial $<$-isomorphism $\varphi: X \to M$, where $X \subseteq M$, for any finite state s over *X* and for any *k*-tuple $\bar{a}$ in *X*.

Definition 2. We will speak that a complete theory *T* has *the Property of Isolation* if there is a cardinal λ such that for any pseudo-finite set *A* and for any element $\bar{a}$ of a model of the theory *T* there exists $A_0 \subseteq A$ such that $|A_0| < \lambda$ and $tp(\bar{a}/A_0)$ isolates $tp(\bar{a}/A)$.

For any subsets *A, B* of a structure *M* we write $A < B$ if $a < b$ every time when $a \in A$ and $b \in B$. If $A \subset M$ and $x \in M$ write $A < x$ if $A < \{x\}$. For an arbitrary complete 1-type *p* we denote by $p(M)$ the set of realizations of the type *p* in *M*. *An open interval I* in a structure *M* is a parametrically definable subset of the structure *M* of the following form:

$$I = \{c \in M: M \models a < c < b\}$$

for the some $a, b \in M \cup \{-\infty, \infty\}$, where $a < b$. Similarly, it is possible to define *closed, semi-ope-semi-closed*, etc. intervals in *M* so that for example any point of the structure *M* is itself a (trivial) closed interval. A subset *A* of structure *M* is *convex* if for any $a, b \in A$ and $c \in M$ every time when $a < c < b$ we have $c \in A$.

The present paper concerns the notion of *weak o-minimality* firstly introduced by *M.* Dikmann in [1]. *A weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure *M* is an union of finitely many convex sets in *M*. Such a structure *M* is *o-minimal* if any definable (with parameters) subset of the structure *M* is an union of finitely many intervals in *M*. Thus, the weak o-minimality is generalization of a o-minimality. The rank of convexity of formula with one free variable has been introduced in [2]. In particular, a theory has rank of convexity 1 if there is no definable (with parameters) relation of equivalence with infinitely many convex infinite classes. Obviously that any o-minimal theory has rank of convexity 1. A. Pillay and Ch. Steinhorn have proved for the o-minimal theory as holding the Exchange Principle for algebraic closure, and density of isolated types [3]. For a weakly o-minimal theory both the first property and the second property do not hold in general. In the present work we introduce the notions of quite orthogonality of 1-types and quite o-minimal theory, and we prove some properties of such a theory. In particular, it is proved that a quite o-minimal theory of finite rank of convexity has the Exchange Principle for algebraic closure, and also some remarks on density of isolated types are represented. As corollary we receive reducibility of expanded queries to limited over a quite o-minimal domain.

Definition 3. [2] Let M be a linearly ordered structure, $\phi(x)$ be an M-definable formula with one free variable.

Rank of convexity of the formula $\phi(x)$ $(RC(\phi(x)))$ is defined as follows:

1) $RC(\phi(x)) = -1$ if $M \models \neg \exists x\, \phi(x)$

2) $RC(\phi(x)) \geq 0$ if $M \models \exists x\, \phi(x)$

3) $RC(\phi(x)) \geq 1$ if $\phi(M)$ is infinite

4) $RC(\phi(x)) \geq \alpha + 1$ if there exists a parametrically definable relation of equivalence $E(x, y)$, such that there are b_i, $i \in \omega$, which satisfy to the following conditions:

- For any $i, j \in \omega$ every time when $i \neq j$ we have $M \models \neg E(b_i, b_j)$
- For any $i \in \omega$ $RC(E(x, b_i)) \geq \alpha$
- For any $i \in \omega$ $E(M, b_i)$ is convex and $E(M, b_i) \subset \phi(M)$

5) $RC(\phi(x)) \geq \delta$ if $RC(\phi(x)) \geq \alpha$ for all $\alpha \leq \delta$ (δ is limiting ordinal). If $RC(\phi(x)) = \alpha$ for the some α, we speak that $RC(\phi(x))$ is defined. Otherwise (i.e. if $RC(\phi(x)) \geq \alpha$ for all α) we assume $RC(\phi(x)) = \infty$.

In the following definitions M is a weakly o-minimal structure, $A, B \subseteq M$, M – $|A|$ +-saturated, $p, q \in S_1(A)$ – non-algebraic.

Definition 4. [4] We shall speak that the type p is not *weakly orthogonal* to type q if there exists A-definable formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

Lemma 5. ([4], Corollary 34 (iii)) The relation of non-weak orthogonality is the relation of equivalence on $S_1(A)$.

Definition 6. [5] We shall speak that the type p is not *almost orthogonal* to type q if there exists A-definable formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $H(M, \alpha) \neq \varnothing$ and $\beta_1 < H(M, \alpha) < \beta_2$.

Fact 7. Non-almost orthogonality of 1-types implies its non-weak orthogonality.

Definition 8. [5] We shall speak that a weakly o-minimal theory T is *almost o-minimal* if the notions of weak and almost orthogonality of 1-types coincide.

Fact 9. Any o-minimal theory is almost o-minimal.

Example 10. Let $M = \langle M, =, <, U_1^1, U_1^2, R^2 \rangle$, where $\langle M, < \rangle$ has the order type Q. The universe M is non-crossing union of U_1 and U_2 with a condition $a < b$ every time when $a \in U_1$, $b \in U_2$, and each predicate U_i has no endpoints in M. In order to define R we shall identify U_i with Q for everyone $i \leq 2$, and for anyone $a \in U_1$ and $b \in U_2$ we have $R(a, b) \Leftrightarrow b < a + \sqrt{2}$.

It is not difficult to prove that $Th(M)$ is a weakly o-minimal theory. Let $p := \{U_1\}$, $q := \{U_2\}$. It is easy to understand that $p, q \in S_1(\varnothing)$, the type p is almost orthogonal to the type q, but the type p is not weakly orthogonal to the type q, i.e. $Th(M)$ is not almost o-minimal.

Definition 11. We shall speak that the type p is not *quite orthogonal* to type q if there exists A-definable bijection $f: p(M) \to q(M)$.

Definition 12. We shall speak that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality of 1-types coincide.

Fact 13. Any o-minimal theory is quite o-minimal.

Fact 14. Any quite o-minimal theory is almost o-minimal.

Fact 15. The relation of non-quite orthogonality is the relation of equivalence on $S_1(A)$.

Example 16. [6] Let $M = \langle M, =, <, P^1, f^1 \rangle$. Here P is a unary predicate and f is a unary function with $Dom(f) = \neg P$, $Ran(f) = P$ (therefore, formally, M is 2-sorted). The universe of the structure M is non-crossing union of P and $\neg P$, where $x < y$ every time when $x \in P$ and $y \in \neg P$. In order to define f we shall identify P with Q (where Q is the order of rational numbers) and $\neg P$ with $Q \times Q$ (which it is lexicographically ordered), and for any $m, n \in Q$ let $f(m, n) = n$.

It is not difficult to prove that Th (M) is an almost o-minimal theory. Let $p: = \{\neg P\}$, $q: = \{P\}$. It is possible to understand, that $p, q \in S_1(\varnothing)$, the type p is not almost orthogonal to type q, but at the same time the type p is quite orthogonal to the type q, i.e. $Th(M)$ is not quite o-minimal. It is obvious, that the Exchange Principle for algebraic closure do not hold in M.

Example 17. Let $M = \langle M, =, <, U_1^1, U_1^2, E^2, R^2 \rangle$, where $\langle M, < \rangle$ has the order type Q. The universe M is non-crossing union of U_1 and U_2 such that $a < b$ every time when $a \in U_1$, $b \in U_2$, and each predicate U_i has no endpoints in M. In order to define E and R, we shall identify U_i with $Q \times Q$ (which it is lexicographically ordered) for every $i \le 2$, so for any $(a_1, c_1), (a_2, c_2) \in U_1$ we have $E((a_1, c_1), (a_2, c_2)) \Leftrightarrow a_1 = a_2$ and for any $(a, c) \in U_1$ and $(b, d) \in U_2$ we have

$$R((a, c), (b, d)) \Leftrightarrow b < a \vee [b = a \wedge d < c + \sqrt{2}].$$

It is not difficult to prove that $Th(M)$ also is almost o-minimal. Let $p: = \{U_1\}$, $q: = \{U_2\}$. It is easy to understand that the type p is not almost orthogonal to the type q, but at the same time the type p is quite orthogonal to the type q, i.e. the given theory also is not quite o-minimal. We shall notice that the given theory has a trivial algebraic closure and consequently the Exchange Principle for algebraic closure holds in any model of the given theory.

Lemma 18. Let M be a linearly ordered structure, $Th(M)$ is $\aleph_0$-categorical, $A \subseteq M$, $a, b \in M$, $a \ne b$. Then if $a \in dcl(A \cup \{b\}) \setminus dcl(A)$ we have $tp(a/A) \ne tp(b/A)$.

Proof of the Lemma 18.

As $a \in dcl(A \cup \{b\})$ there is a formula $\phi(x, y, \bar{c})$, $\bar{c} \in A$, such that $M \models \phi(b, a, \bar{c}) \wedge \exists! y\, \phi(b, y, \bar{c})$. If we assume that $tp(a/A) = tp(b/A)$ then we have $dcl(b, \bar{c})$ is infnite contradicting the $\aleph_0$-categoricity of $Th(M)$. □

Lemma 19. Let M be a linearly ordered structure, $Th(M)$ is $\aleph_0$-categorical, $a, b, \bar{c} \in M$, $a \in dcl(b, \bar{c}) \setminus dcl(\bar{c})$. Then $b \in dcl(a, \bar{c})$ if and only if $tp(a/\bar{c})$ is not quite orthogonal to $tp(b/\bar{c})$.

Proof of the Lemma 19.

($\Leftarrow$) Let $p = tp(b/\bar{c})$, $q = tp(a/\bar{c})$. As the type p is not quite orthogonal to the type q there exists $\bar{c}$-definable bijection $f: p(M) \to q(M)$. We confirm that $f(b) = a$, we shall have whence that $b \in dcl(a, \bar{c})$. We assume the contrary: $f(b) \ne a$, i.e. exists $a' \in q(M)$ such that $a' \ne a$ and $f(b) = a'$. Then we receive that $a \in dcl(a', \bar{c})$, contradicting the Lemma 18. □

Corollary 20. Let T be an $\aleph_0$-categorical quite o-minimal theory. Then in any model of theory T the Exchange Principle for algebraic closure holds.

Proposition 21. Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ – non-algebraic. We shall assume that there exist $a, b \in p(M)$ such that $a \ne b$ and $a \in dcl(A \cup \{b\})$. Then if $b \notin dcl(A \cup \{a\})$ then T has an infinite rank of convexity.

Proof of Proposition 21.

As $a \in dcl(A \cup \{b\})$ there is an A-definable formula $\phi(x, y)$ such that $M \models \phi(b, a) \wedge \exists! y\, \phi(b, y)$. As $b \notin dcl(A \cup \{a\})$, $\phi(M, a)$ is infinite and by weak o-minimality of T we can assume that $\phi(M, a)$ is convex. We shall consider the following formula:

$$E_1(x_1, x_2): = \exists y\, [\phi(x_1, y) \wedge \phi(x_2, y)].$$

Obviously that E_1 is the relation of equivalence partitioning $p(M)$ on infinite number of infinite convex classes. Further for every natural $n \ge 2$ we shall consider the following formula:

$$E_n(x_1, x_2): = \exists y_1 \exists y_2\, [\phi(x_1, y_1) \wedge \phi(x_2, y_2) \wedge E_{n-1}(y_1, y_2)].$$

Also it is easy to understand that for every natural $n \ge 2$ E_n is the relation of equivalence breaking each class of equivalence under the relation E_{n-1} on infinite number of classes, and each class is convex and open. Thus, we receive that the given theory has an infinite rank of convexity. □

Remark 22. The given result is similar to the known result from the theory of stability: if *T* is a ω-stable theory under other conditions of Proposition 21 *T* has infinite rank of Morli.

Theorem 23. Let *T* be a quite o-minimal theory of a finite rank of convexity. Then in any model of theory *T* the Exchange Principle for algebraic closure holds.

Proof of the Theorem 23.

We assume the contrary: there is a model *M* of the theory *T* in which the Exchange Principle for algebraic closure doesn't hold. Then there exist $a, b, \bar{c} \in M$ such that $a \in dcl(b, \bar{c}) \setminus dcl(\bar{c})$, but $b \notin dcl(a, \bar{c})$.

Let $p := tp(a/\bar{c})$, $q := tp(b/\bar{c})$. If $p = q$ then by Proposition 21 *T* has an infinite rank of convexity. Hence, $p \neq q$. As $a \in dcl(b, \bar{c})$, then the type *p* is not weakly orthogonal to the type *q*, and by the quite o-minimality of *T p* is not quite orthogonal to *q*. Hence, there exists $\bar{c}$-definable bijection $f: q(M) \to p(M)$. If $f(b) = a$ then $b \in dcl(a\ \bar{c})$, contradicting our assumption. If $f(b) = a' \neq a$ then $a \in dcl(a', \bar{c})$, and $tp(a/\bar{c}) = tp(a'/\bar{c})$. If $a' \notin dcl(a, \bar{c})$ then *T* has an infinite rank of convexity. Hence, $a' \in dcl(a, \bar{c})$, whence $b \in dcl(a, \bar{c})$, again contradicting our assumption. □

Corollary 24. Let *T* – an $\aleph_0$-categorical quite o-minimal theory. Then *T* has the Exchange Principle for algebraic closure.

Theorem 25. [7] Any $\aleph_0$-categorical quite o-minimal theory is binary.

Theorem 26. [8] Suppose that the theory of the first order of structure of *M* has the Isolation Property. Let an expanded query φ be locally generic over finite states. Then φ is equivalent over finite conditions to a limited query.

Theorem 27. [8] Let *T* be a quasi-o-minimal theory. Then *T* has the Isolation Property.

Theorem 28. Let *T* – an $\aleph_0$-categorical quite o-minimal theory. Then *T* has the Isolation Property.

Corollary 29. Let *T* – an $\aleph_0$-categorical quite o-minimal theory. Then any expanded query being locally generic over finite states is equivalent to a limited query.

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